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MECHANISM

FUNDAMENTAL THEORY OF THE MODIFICATION AND TRANSMISSION OF MOTION

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PREFACE

In the production, transmission, and use of power, the machinery involved must consist of machine elements which are so designed that:

1. They will function together properly in producing the desired mechanical movements.

2. They will be able to stand up without rupture or undue distortion under the forces to which they are subjected.

Hence the design of machinery requires an understanding of the subjects of

1. *Mechanism,* which deals with mechanical movements and the combinations of links or machine elements by which these movements are effected.

2. *Machine Design,* which imparts the information that permits each machine element to be so proportioned that it will safely resist or withstand the external force or load to which it is subjected.

This text on the subject of Mechanism is therefore offered as a companion text to the author's *Machine Design* in order to cover more completely this interesting field.

Since there are many elemental mechanisms, each composed of links which may take various forms, and since these elemental mechanisms can be brought together into more complicated or compound mechanisms, a great deal of descriptive matter could be introduced into a text on this subject. However, the author has limited such purely descriptive matter as much as possible and has emphasized the fundamental theory of the subject. The basic principles of the three ordinary modes of transmission of motion are developed, and a limited number of types of mechanisms of each mode are studied. In analyzing the relative motion of the elements of these mechanisms, the graphical method of analysis, rather than the algebraic method, is stressed. The text includes many graphical constructions which are fully explained, and many illustrative examples with their solutions. The text material is presented in such a way as to help the student develop an orderly process of thinking and an ability to analyze and visualize that will permit him to cope with other kinematical problems.

The student is advised to study each chapter carefully and then solve each graphical construction and example, and prove any theorem, statement, or formula given therein, without the aid of the text or referring to it only at such times as he needs help. Having done this, he should answer every question and solve every problem at the end of the chapter before proceeding to the next.

The presentation of the text material is based on the assumption that the mathematical training of the student has extended through an elementary course in trigonometry and that he is well informed in the subject of mechanical drawing. A knowledge of the mechanics of physics, while not absolutely necessary to the understanding of this subject, is of course a distinct advantage.

It was the aim of the author to so arrange and present the subject matter of this text that it may be readily assimilated without help from any other source.

The author wishes to express his appreciation to the various manufacturers who were so generous in supplying information and illustrations from their catalogs and bulletins. The student's attention is called to the great fund of information relative to this subject that may be obtained from such industrial sources; it is mainly descriptive and is particularly instructive after one has gained a fundamental knowledge of the subject. Information and data for this text have been drawn from other sources as well, and a sincere effort has been made to acknowledge this courtesy throughout the text. Appreciation is also due Mr. Robert C. Williams for work on the illustrations, and Mrs. F. Swarts for editorial work on the manuscript.

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RELATIVE MOTION

Courtesy of Link-Belt Company, Chicago, Illinois

MECHANISM

CHAPTER I

FUNDAMENTAL CONSIDERATIONS

Introduction. Mechanism, or Kinematics of Machinery, is that subject which deals only with the relative motion of machine parts, often referred to as machine elements or links. It gives no consideration to that phase of machine design which inquires into the forces involved and which creates machine elements of the proper size and shape to withstand safely those forces or loads to which they are subjected. The latter is the province of the subject known as Machine Design, which therefore creates machine parts which are resistant or rigid; that is, they stand up under their loads without undue distortion or change of shape. In the study of Mechanism, the machine parts are assumed to be resistant. Mechanism, when so defined, is called Pure Mechanism.

A Mechanism. A group of several machine elements so shaped and arranged that the motion of one element will compel or necessitate a certain definite motion for each of the others, is known as a *Mechanism*. It will be seen from this definition that the only office of a mechanism is to modify and transmit motion.

A Machine. A *Machine* is a combination of interrelated bodies, links, or elements, which are characterized by being resistant and, in general, rigid. These links are so chosen and arranged that, in operation, they can collectively either transform other forms of energy into mechanical energy, or they can receive mechanical energy (power) from some external source, transmit, and modify it, thus doing work in performing some task.

From this definition, it is evident that a machine must be made up of a combination of mechanisms, each of which has a certain part to play or duty to perform. However, the whole combination of mechanisms is designed to work together in order that some desired result may be accomplished. Take for example the recipro-

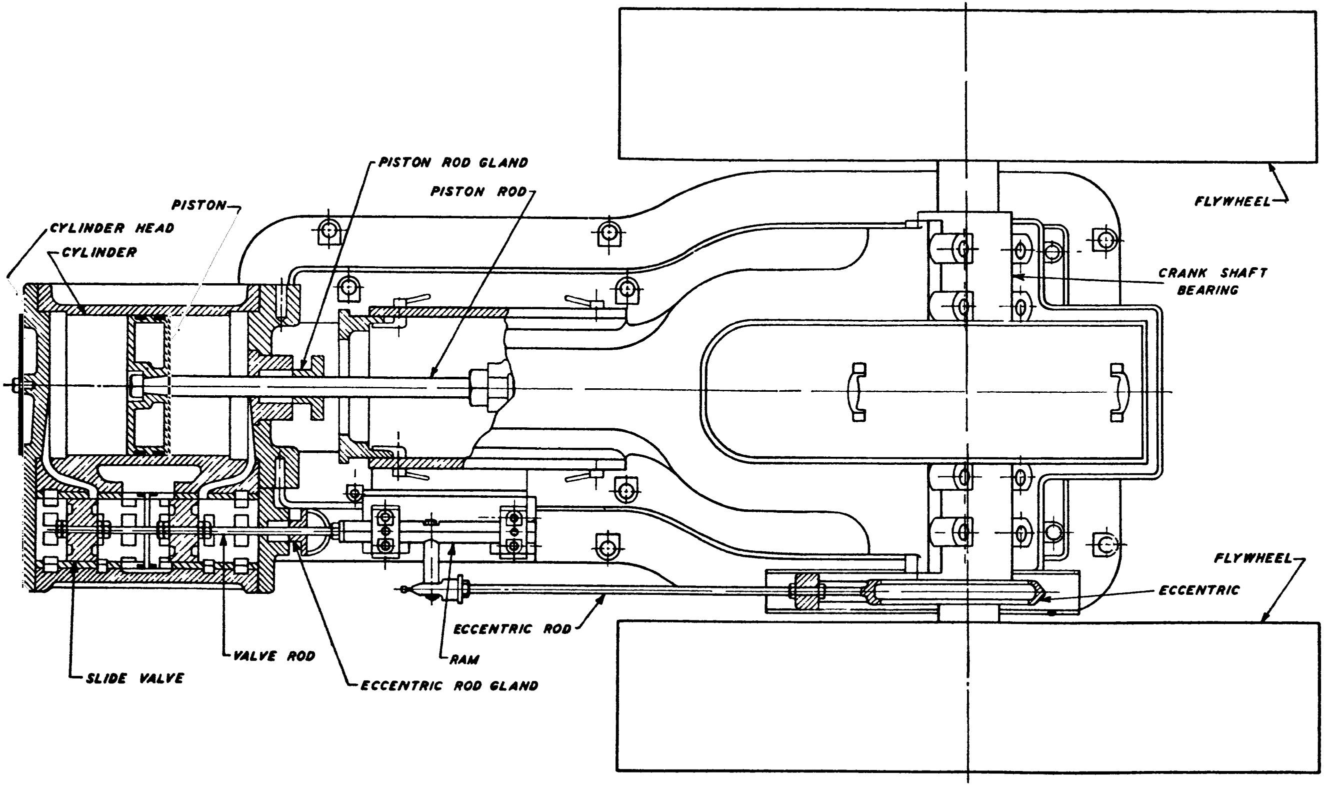


Fig. 1A. Plan View of Center-Crank, Slide-Valve Steam Engine

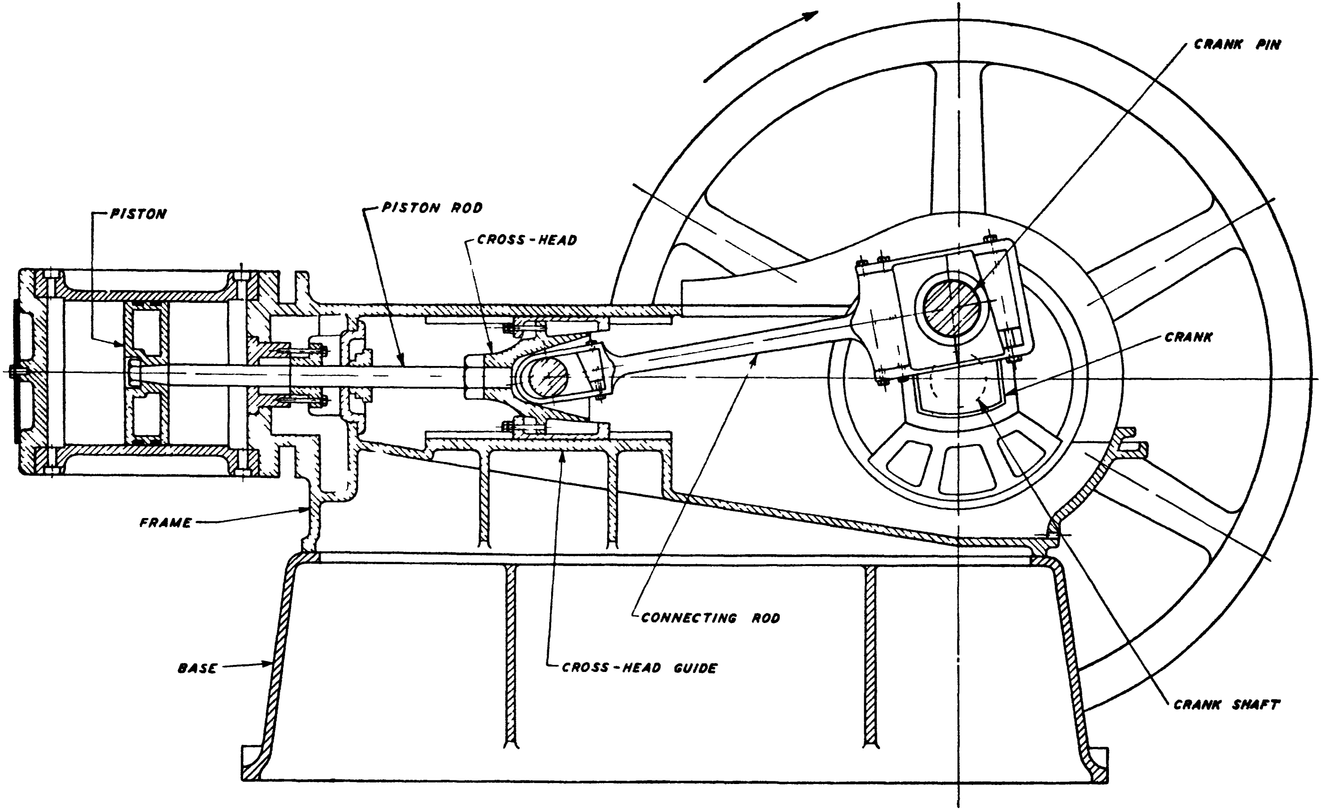


Fig 1B Side View of Center Crank, Slide-Valve Steam Engine

cating steam engine, Fig. 1, which employs a slide valve to open and close its ports. The major mechanism is of the type shown in Fig. 2. In the main, it consists of the piston, piston rod, crosshead, connecting rod, crank, and crank shaft. By these links the rectilinear motion (translation) of the piston is changed into the rotation (revolving) of the crank shaft. The slide valve must be caused to slide back and forth over the ports in order to open and close them at the proper time. This is brought about by a mechanism similar to the major mechanism, and it is known as a valve gear. See Fig. 3.

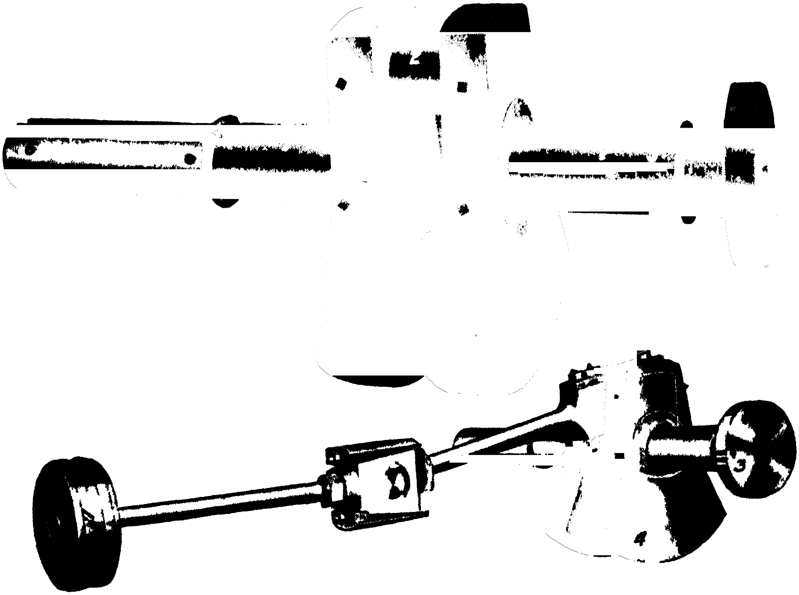
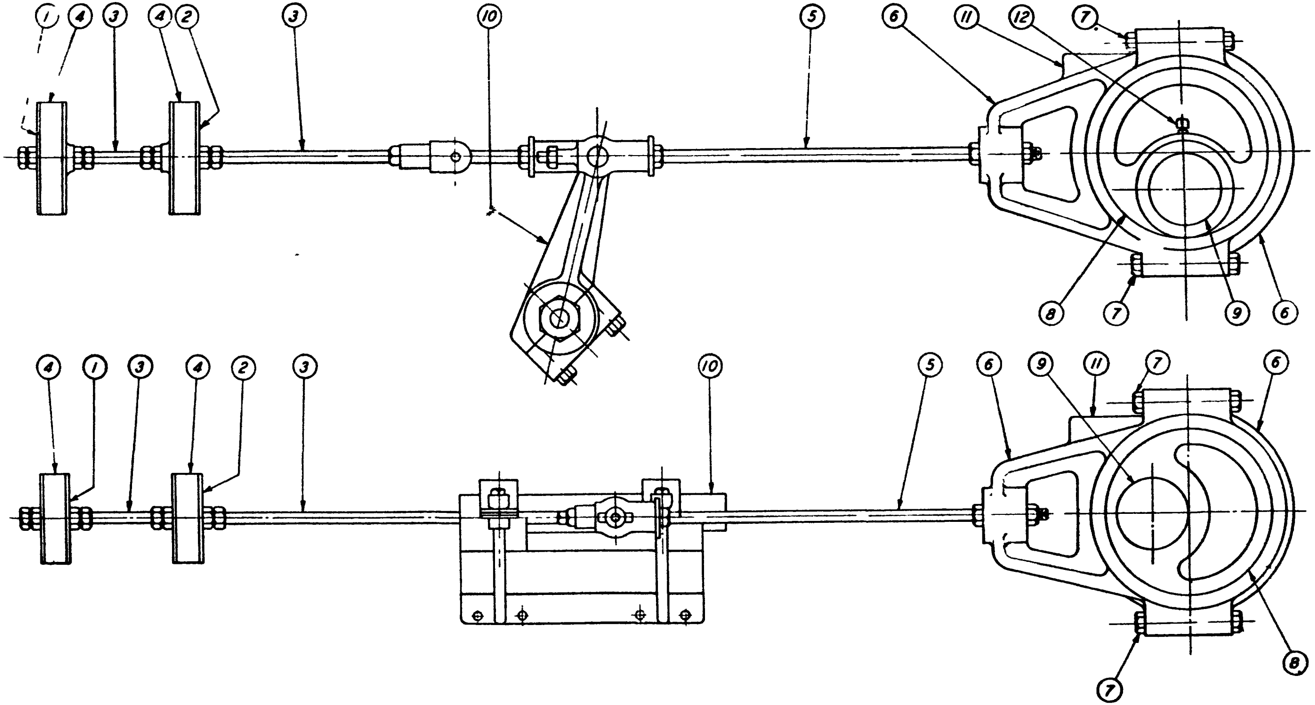


Fig 2 Connection of Piston, Crosshead, Connecting Rod, and Crank Shaft

The chief links of the valve gear are the valve, valve rod, rocker arm or ram, eccentric rod, and an eccentric which is keyed to the crank shaft and hence has the rotative motion of the latter. The rotation of the crank shaft undergoes a modification, through the valve-gear mechanism, that results in the desired movement of the slide valve.

Still another mechanism of this engine may be one to drive a fly-ball governor (see Fig. 4) and thus regulate the speed of the engine. All of the individual mechanisms, each having a specific



1. Piston Slide Valve
2. Piston Slide Valve
3. Valve Rod
4. Packing Rings
5. Eccentric Rod
6. Eccentric Strap
7. Strap Bolts
8. Eccentric
9. Crank Shaft
10. Rocker Arm and Ram (below)
11. Oil Cup
12. Set Screw

Fig. 3. Eccentric Mechanism Showing Rocker and Ram Methods of Connecting Eccentric Rod with Valve Rod

part to play, are properly synchronized and, together, form the machine.

Frame. In every series of links constituting a mechanism or a machine, one link is either fixed (stationary) or may be so considered. It is fixed, in that it is assumed to be securely fastened to the earth and hence has no motion with respect to the earth. This fixed link of a machine is known as the *Frame.* It is the supporting structure for all of the moving links, and plays a most important part in defining their motions. The motion of any machine part is generally given or determined with respect to the frame. Since a machine can have but one stationary part, the frame of the machine is the fixed link for each mechanism that enters into its design. From a kinematic or mechanism standpoint then, the frame or stationary link of the steam engine, Fig. 1, consists not only of the frame proper, but all other stationary or fixed parts as well, such as the base, cylinder, cross-head guides, bearings, etc. The frame of a locomotive or of an automobile is kinematically considered in the same manner as the frame of a stationary machine, even though the former has motion with respect to the earth.

Fig. 4. Fly-Ball Governor
Courtesy of Gardner-Denver Company, Quincy, Illinois

Constrained Motion. Each link of a mechanism must have

Constrained Motion; that is, any link must be so guided by the other links to which it is attached or with which it is in contact, that any point of it must move in a certain definite path. Thus in Fig. 1 the steam cylinder constrains the motion of the piston in that it permits the piston to have motion only along its axis; the crosshead guides constrain the crosshead so that it must move along their bearing surfaces; the crank compels the crank pin to take a circular path, etc.

In Fig. 5 five links are joined together by pinned joints. The hatch lining below link *A* designates it as the fixed link or frame. This convention will be followed throughout the text. An inspection of this figure clearly indicates that although the links are joined together, there is no assurance that a point such as *M* of link *E* will move in a certain designated path. In fact there are several paths which point *M* may follow, for links *B* and *C* can move, with respect to each other, either oppositely or in the same direction, no control of one over the other being possible from the combination of links as shown. This is a condition of only partial constrainment at the most, and such a series of links is an Unconstrained Chain and hence is not a mechanism.

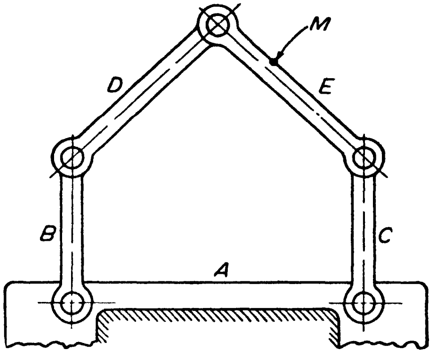


Fig. 5 An Unconstrained Chain

Fig. 6 shows a chain consisting of four links, *A*, *B*, *C*, and *D*, connected by pin joints. This chain is known as the four-bar linkage. Any movement of link *A* about its pin joint, with respect to the fixed link *D*, will cause every point of the links *B* and *C* to have a motion along a definite path. When such a condition exists, the chain gives full constrainment of motion. Such a chain, having one fixed link and giving complete constrainment of motion, is a mechanism. Whether or not an ordinary chain is a mechanism (that is, one which gives full or complete constrainment) can be told as a rule by inspection.

Links. Although the term "link" has been used in previous discussions, it should be considered more fully. A link is any machine part which connects other parts that have motion relative to it. It may be a fixed link, or frame, which merely supports other links. It may serve in the capacity of a guide; for example, the cylinder of Fig. 1 guides the piston. It may transmit motion, as the connecting rod of Fig. 1, in which case the motion of the crank is produced by it from the motion it receives from the crosshead (or piston). The term "link" is so general that it may be used in reference to a gear, pulley, or cam. In addition to these, the following types of links are so commonly used that they should be defined.

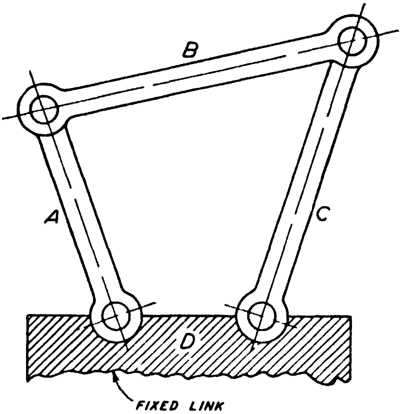


Fig. 6. A Constrained Chain or a Mechanism

In general, a *Crank* may be defined as an arm which makes complete turns; in other words, a crank rotates about a fixed center. It sometimes takes the form of a disk or an eccentric. See Figs. 1, 2, links *A* and *C* of Fig. 8, and link *A* of Fig. 9.

"Lever" is a term that in many cases is used in the place of the term "crank," particularly where the crank oscillates or reverses its direction of rotation after passing through a limited central angle with respect to its fixed center or pivot. The term "lever" is also applied to that type of link which is in reality composed of two cranks which are integral and have the same hub. An example of this is the bell-crank lever. The cam follower *B* of Fig. 7 could be called a lever.

A *Slider* or *Sliding Block* is a link that slides along the curved or plane surface of another link. See Figs. 1, 2, 3, and 9 which illustrate such links.

All the links to which reference has been made have been *Rigid Links;* that is, they can be subjected to a push (thrust) or a pull (tensile load). Other links are *Flexible.* To this classification belong all so-called wrapping connectors, such as flat belts, V-belts, ropes, and chains. Since only a pull can be registered by the latter, they are known as *Tension Links.* Upon occasion, even a fluid used to transmit a pressure may become a link of a mechanism; an example of this is the use of oil in the hydraulic braking system of an automobile.

It is often the case that several machine parts can be considered as a single link from a kinematic point of view. Thus in the steam engine of Fig. 1, the piston, piston rod, and crosshead are different machine parts, but, being rigidly attached to each other (see Fig. 2), it is conceivable that they might be constructed as a single link. It is also evident that all of these links have the same motion. Under these conditions, such a series of machine parts becomes kinematically a single link, and in the diagrams that follow in this text they will be so represented. See Fig. 9. The crank and crank shaft are another example of two machine parts which are in reality one kinematic link. See Fig. 2.

Driver and Follower. That link of a mechanism which causes motion is called the *Driver* and the member to which motion is transmitted is called the *Follower.* In a mechanism like the four-bar linkage of Fig. 6, the first driver, *A,* drives the link *B,* to which it is connected or with which it is paired; and link *B* in turn drives link *C,* to which it is connected. Thus link *B* is in a sense both a driver and a follower. However, in this mechanism, *A* is considered the driver and *C* the follower, and *B* is called the connector or connecting rod. A *Train of Mechanism* has several actual drivers, each having its own follower. See Figs. 150 and 151.

Higher and Lower Pairing. Machine parts may be in contact with each other in any one of three different ways, namely:

(*a*) at a point
(*b*) along a line
(*c*) over a surface

As examples of these, there may be cited: (*a*) the point contact of a ball bearing or of meshing helical gear teeth; (*b*) the line contact of a roller bearing, of meshing spur, or of meshing bevel gear teeth; (*c*) the surface contact of a plain journal bearing, of a piston with its cylinder, or between a screw and its nut.

In general, pairs are those connections which permit relative motion between the links of a mechanism. A connection in which either point or line contact occurs between the surfaces of the contacting links is referred to as a *Higher Pair,* while a connection in which surface contact occurs is referred to as a *Lower Pair.* Two links of a mechanism at whose connection either higher pairing or lower pairing is maintained are often called a *Kinematic Pair.* Thus in Fig. 6, any pair of links such as B and C are a kinematic pair; and since a pin joint is used as their connection, the cylindrical surfaces of the pin and its bearing yield surface contact and hence lower pairing. The *Elements* of the pair are the surfaces which contact each other and hence work together.

Turning and Sliding Pairs. The pairs used in the connection of the links of Fig. 6 permit of only relative rotation. Such pairs are known as *Turning Pairs.* The pairs used in the case of the cylinder and piston, and the crosshead and its guides of Fig. 1, are surfaces so formed that only relative sliding is permitted during the transmission of motion by the mechanism. Such pairs are called *Sliding Pairs.*

Incomplete Pairing. The pairs so far discussed have been *Complete Pairs;* that is, the form or shape of each element of the pair, together with the rigidity of the material used in the links, produce complete constrainment of the motion of one element by the other without the help of any outside agency or force.

Sometimes an outside agency or force can be utilized, resulting in the simplification of the mechanism in general, or in the construction of one or more links thereof. Let us consider the three-link cam mechanism of Fig. 7. Here a link, A, called a cam, rotates or revolves with respect to the fixed link or frame, *C*. Link *A* is a right circular cylinder in form, but it does not rotate about its geometric axis. Therefore, during its rotation it causes its follower, link *B*, to vibrate or oscillate (partial rotation) about its axis with respect to the frame *C*. While *B* is moving upward, it is under the complete control of the cam; but as it moves downward, it is the action of the force of gravity (that is, the weight of *B*) that keeps the follower in contact with its cam or driver. If the force of gravity should meet with some force that would resist or oppose it, such as a frictional resistance between the bearing surfaces (the elements of the pair) of links *B* and *C*, link *B* would stay in its raised position and the mechanism would be ineffective. Of course the bearing surfaces between *B* and *C* would be lubricated so that this should not occur. An extension spring also could be introduced between links *B* and *C* to keep *B* in contact with *A*. When a condition like this occurs in a chain of links, *Incomplete Pairing* is said to exist between the two links involved. However, the chain is a mechanism in every respect.

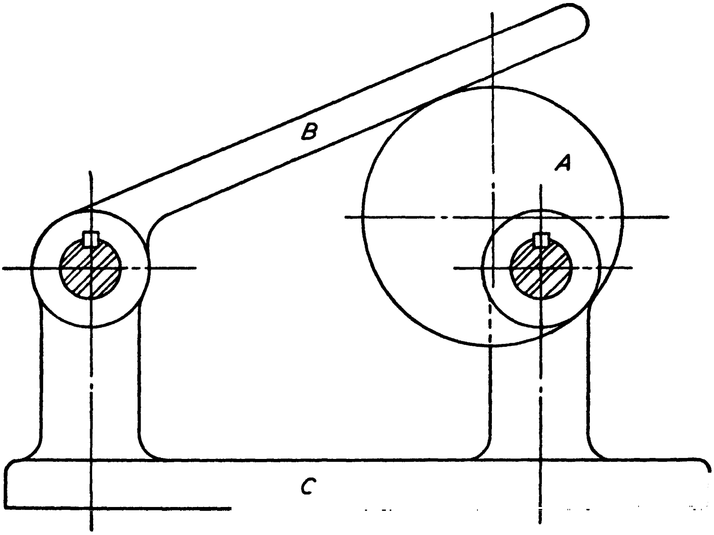


Fig. 7. Cam Mechanism Showing Incomplete Pairing

Graphic Solutions. The problems that are presented in Mechanism can be solved by both graphical and analytical or mathematical methods. Some solutions that would become rather involved and tedious if brought about mathematically, are effected directly and rather easily by a graphical method. If care is taken and good drawing instruments are used, the results obtained on the drawing board are sufficiently accurate for all practical purposes. In this volume both methods will be used, the one selected depending on the type of problem.

Skeleton Diagrams or Layouts. In preparing a problem for a graphical solution, it is unnecessary and too laborious to use a projection drawing of the mechanism one is to work with; therefore the projection drawing is replaced by a *Skeleton Diagram* or *Layout*, in which the various links of the mechanism are represented by lines. In linkages with pin joints, these lines are in a sense center lines of the links, as they are drawn from the center of the pin joint at one end of the link to the center of the joint at the other end. The pin joint will be represented by a very small circle, the center of which is the center of the joint or the projection of the axis of the pin of the joint. It is also the intersection of a pair of lines representing the two links having a connection at that pin joint.

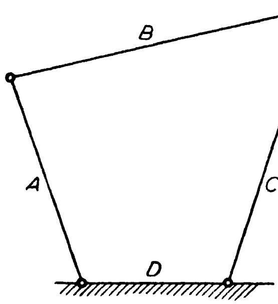


Fig. 8 Skeleton Diagram of Four-Bar Linkage

In all skeleton diagrams, the frame or fixed link will be designated by hatch lines and there will be no attempt to present it in outline. Hence wherever hatch lines appear they represent the one and only fixed link of that particular mechanism. A sliding link (called a slider or a sliding block) like the cross-head of Figs. 1 and 2, will be represented by a small rectangle with lines drawn along each side to show the constraining link and to indicate the direction of motion of the slider. In every case, the name of the link will be a capital letter placed somewhere along the line representing that link (the line of the link). The letter designating the frame is generally given but once, even though the frame shows on the diagram in several places.

A skeleton diagram of the linkage of Fig. 6 is given in Fig. 8 and the steam engine mechanism which is illustrated in Fig. 2 is given as a skeleton diagram in Fig. 9. In the latter, *A* is the crank, *B* is the connecting rod, *C* is the cross-head (combined with the piston rod and piston), and *D* is the fixed link or frame. It should be noted in studying Figs. 8 and 9 that, since a link is represented as a line, a point can lie on a link and still not be on the line of that link in the diagram. To make this clear, point *L* in Fig. 8 is a point on link *C*. Such a point will be represented always by a dot and will be connected to the link to which it belongs by a perpendicular line drawn from the point to the line of the link.

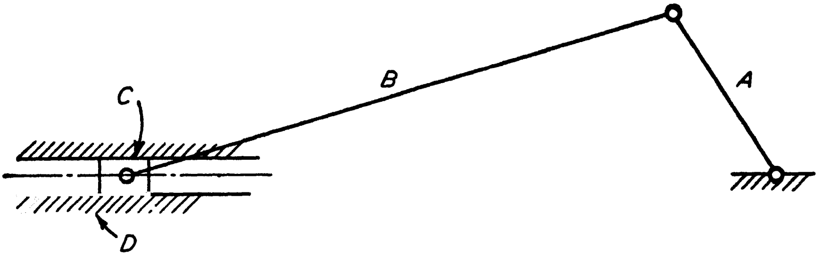


Fig 9 Skeleton Diagram of the Steam Engine Mechanism

QUESTIONS

1. Of what does the subject of Mechanism treat?
2. Of what does the subject of Machine Design treat?
3. Define the term *mechanism.*
4. Define the term *machine.*
5. In what different ways does a link of a mechanism function?
6. Is one link of a mechanism always fixed or stationary?
7. Define the following terms: *frame, crank, lever,* and *slider.*
8. Contrast *lower pairing* and *higher pairing* and give examples of each.
9. What is a *turning pair?*
10. What is a *sliding pair?*
11. Name the various links that enter into the two major mechanisms of simple slide-valve steam engine.

12. What is meant by *constrained motion?*

13. Name a series of related machine parts that are considered as a single link from the standpoint of kinematics of machinery.

14. What are the *elements* of a pair?

15. Is it possible for *incomplete pairing* to exist in a mechanism?

CHAPTER II

MOTION

Motion and Rest. When a body changes its position, it is said to have *Motion* or it is said to be in motion. On the other hand, if a body does not change its position, it is said to be at *Rest.*

These definitions of motion and rest, while quite sufficient as to the terms themselves, might give rise immediately to the following questions: With respect to what is the body in motion? With respect to what is the body at rest? For it is quite conceivable that a body might be in motion and at rest at the same instant, but with respect to two different bodies. For instance, a trunk lies on the floor of a moving baggage car. It is evident that the trunk is at rest, or has no motion, with respect to the car, but that it is in motion with respect to the earth. In fact, since the trunk is at rest with the baggage car, it has the same motion with respect to the earth as the car itself. Now if the baggage man decides to shove the trunk across the floor of the moving car, the trunk will have one motion with respect to, or relative to, the car and another motion relative to the earth.

The preceding discussion demonstrates the idea that all motions, as conceived kinematically, are Relative, and that the change of position of a body must be considered with respect to another body which is either at rest or so assumed. Although at times in the study of a mechanism or machine it may be necessary to deal with the relative motion of one member with respect to another which actually is in motion, as a rule the motions referred to are those in relation to the frame. *The use of the term "motion" in this text should be interpreted as a change of position with respect to the frame unless clearly stated otherwise.* It should be noted also that if two bodies, A and B, have relative motion with respect to each other, the relative motion of A with respect to B is the same as the relative motion of B with respect to A.

Path. Several things define or describe the motion of a body or a point thereof. One of these is the path a point of the body is taking. The *Path* of a point is the line which it traces as it changes its position. It may be a straight (right) line or curved. It may or it may not lie in a plane. Since the line is traced by the point, the direction of the line is the direction of the motion of the point. If the line is a straight line, the direction of motion of the point is along the line itself. If the line is a curved line, the direction of motion of the point at any instant is along a tangent to the curve, the tangent being located at the position occupied by the point at that instant. The tangent then evidently indicates or gives the direction of motion which the point has at that instant, or the direction of its instantaneous motion. For example, if the point *A* of Fig. 10 is moving in a circular path, the direction of its motion when in the position shown is the line *AB*. In the same way in Fig. 11, if the point *A* is moving in the curved path, its direction when in the position indicated is given by the line *AB*. An arrow is placed at *B* more clearly to define the motion; that is, to show that it is directed from *A* toward *B* rather than from *B* toward *A*. Of course, the direction of motion of a point which is moving in any path other than a straight line is constantly changing, but the direction in which it is moving at a given instant is the direction which it would take if the forces which constrain it to move in the curved path were removed at that instant.

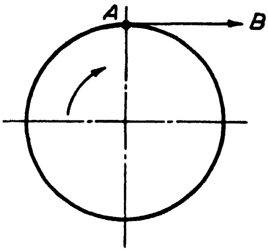


Fig. 10

The path which a point traces may be a closed curve, as that shown in Fig. 10, or a path of limited or finite length, as shown by the line of Fig. 11. In the case of the former, the point may return to its initial position without reversing its path (or direction). In the case of the latter, the point cannot return to its initial position without traversing its path in the reverse direction.

Linear Velocity or Speed. The motion of a point is further described by its *Linear Velocity* (from now on referred to in this volume as *velocity* or *speed*) which is defined as the distance which the point travels in some unit of time. Thus if a point travels 25 feet in one minute, it is said to have a velocity or speed of 25 feet per minute (f.p.m.).

It should be noticed that the unit of velocity is made up of one unit of distance and one unit of time. So, in addition to the unit of feet per minute, other units of velocity are: feet per second (f.p.s.), miles per hour (m.p.h.), etc.

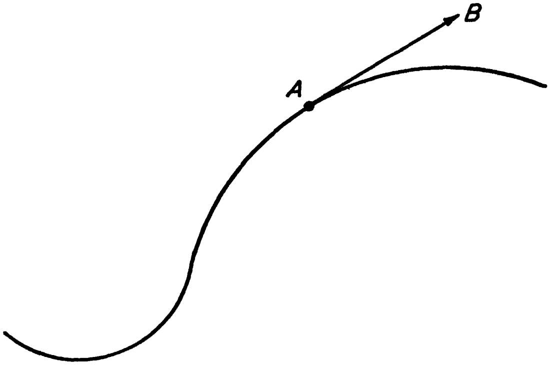


Fig. 11 Direction of Motion of a Point on a Curved Path

Let us suppose that a train travels a distance of 200 miles in 4 hours. It is evident that if the total distance is divided by the time taken in traveling that distance, the answer obtained is the distance traveled in one unit of time, which is (by definition) the velocity. In this case

$$\frac{200}{4} = 50 \text{ miles per hour}$$

which is the velocity of the train. If the distance that the train travels is divided by the time with the time expressed in minutes,

$$\frac{200}{4 \times 60} = \frac{5}{6} \text{ mile per minute}$$

which is the velocity of the train. Also, if the distance the train travels is taken in feet and the time in minutes,

$$\frac{200 \times 5280}{4 \times 60} = 4400 \text{ feet per minute}$$

which is the velocity expressed in still another unit.

In order to obtain the proper unit for the velocity, it should be noted that in each of the above solutions the distance unit of the unit of velocity is the unit used in the total distance covered, while the time unit of the velocity is the unit used in the total time that elapsed while the total distance was covered. Since these results represent the velocity of the same train, they must be expressions of the same velocity, and the conversion from one unit to another could have been made by working directly with the first answer. Thus

$$\begin{aligned} 50 \text{ miles per hour} &= \frac{50}{60} \text{ miles per minute} \\ &= \frac{5}{6} \text{ mile per minute} \end{aligned}$$

or

$$\begin{aligned} 50 \text{ miles per hour} &= \frac{50 \times 5280}{60} \text{ feet per minute} \\ &= 4400 \text{ feet per minute} \end{aligned}$$

If the motion of the train is such that there is no change in its velocity during the 200-mile run, the velocity of 50 miles per hour is its *Uniform* or *Constant Velocity* and the train has a *Uniform Motion.* However, if the velocity of the train is variable, or changes from time to time, the velocity of 50 miles per hour is its *Mean* or *Average Velocity.* Even though the train is slowing down or speeding up, the velocity at any instant may be measured. Such a velocity is an *Instantaneous Velocity.* An instantaneous velocity given as 35 miles per hour implies that the train would travel a distance of 35 miles if it were to continue in motion for one hour after the instant for which its velocity is given. From these statements it is obvious that the instantaneous velocity of a point having uniform motion is constantly the same and moreover is equal to its uniform velocity.

In general, if

V_L = the velocity of a point

S = the space or distance traversed

T = the time in which the distance, S, is traversed,

Then

$$V_L = \frac{S}{T} \tag{1}$$

Multiplying both members of the above equation by T,

$$S = V_L T \qquad (2)$$

Dividing both members of formula (2) by V_L,

$$T = \frac{S}{V_L} \qquad (3)$$

Example 1. Find the average speed or velocity in miles per hour of an automobile that travels a distance of 299 miles in 6½ hours.

Solution. Here $S = 299$ mi. and $T = 6\frac{1}{2}$ hr. Applying formula (1),

$$V_L = \frac{S}{T}$$

and evaluating therein,

$$V_L = \frac{299}{6\frac{1}{2}} = \frac{299}{\frac{13}{2}}$$

$$= 299 \times \frac{2}{13} = 46 \text{ m.p.h.} \quad \textit{Ans.}$$

Example 2. What is the velocity in feet per minute of the automobile of the preceding example?

Solution. Since 1 mi. = 5,280 ft., and 1 hr. = 60 min.,

$$46 \text{ m.p.h.} = \frac{46 \times 5280}{60} \text{ f.p.m.}$$

$$= 4048 \text{ f.p.m.} \quad \textit{Ans.}$$

Example 3. In an automobile engine the length of stroke, or the distance traveled by the piston in going from one end of the cylinder to the other, is 4½ inches. What is the speed of the piston in feet per minute if the crank shaft is making 3,000 revolutions per minute?

Solution. Since there are two strokes of the piston for each revolution of the crank shaft, the piston will make

$$2 \times 3000 = 6000 \text{ strokes per min.}$$

Therefore the distance traveled by the piston in one minute, which by definition is its velocity, is

$$6000 \times \frac{4\frac{1}{2}}{12} = 2250 \text{ f.p.m.} \quad \textit{Ans.}$$

Example 4. A belt travels 6,000 feet in 2½ minutes. Required, its linear velocity or speed in feet per minute.

Solution. Here $S=6{,}000$ ft. and $T=2\frac{1}{2}$ min. Using formula (1),

$$V_L=\frac{S}{T}$$

and evaluating therein,

$$V_L=\frac{6000}{2\frac{1}{2}}=\frac{6000}{\frac{5}{2}}$$

$$=\tfrac{2}{5}\times 6000=2400 \text{ f.p.m.} \quad \textit{Ans.}$$

Acceleration. The rate of change of velocity is called *Acceleration* (in reality, linear acceleration). If a velocity changes from 10 feet per second to 20 feet per second and the change is produced in one second, the acceleration is the change in velocity $(20-10)$ feet per second, in one second; which is written or stated as 10 feet per second per second, and which is abbreviated as 10 ft. per sec.^{2} Likewise, if a velocity changes from 100 feet per minute to 200 feet per minute in 4 minutes, the acceleration is $\frac{200-100}{4}$ or 25 feet per minute per minute (25 ft. per min.^{2}). It is evident that if the increase in velocity is not constant, such an acceleration is the mean or average acceleration. Acceleration is evidently positive when the velocity of a point is increasing; it is negative when the velocity is decreasing; and it is equal to zero when the velocity is uniform or constant. Zero acceleration of course means that there is no acceleration. An acceleration can be expressed in any unit of velocity per any unit of time.

In general, if

A_c = the acceleration of a point
V_{L1} = velocity at the beginning of an interval of time
V_{L2} = velocity at the end of the interval of time
T = the interval of time

then, by definition,

$$A_c=\frac{V_{L2}-V_{L1}}{T} \tag{4}$$

Multiplying both members of the equation by T,

$$A_cT=V_{L2}-V_{L1}$$

Transposing,

$$V_{L2}=V_{L1}+A_cT \tag{5}$$

Example 1. Find the acceleration of a belt whose velocity changes from 2,000 feet per minute to 3,000 feet per minute in a period of time of 4 minutes.

Solution. Here $V_{L2}=3{,}000$ f.p.m., $V_{L1}=2{,}000$ f.p.m. and $T=4$ min. Applying formula (4),

$$A_c=\frac{V_{L2}-V_{L1}}{T}$$

and evaluating,

$$A_c=\frac{3000-2000}{4}=\frac{1000}{4}$$

$$=250 \text{ ft. per min. per min.} \quad \textit{Ans.}$$

Example 2. A point has an initial velocity of 10 feet per second. If its average acceleration during an interval of time of 30 seconds is 2 feet per second per second, what is its velocity at the end of this interval of time?

Solution. Here $V_{L1}=10$ f.p.s., $T=30$ sec., and $A_c=2$ ft. per sec.^{2}. Applying formula (5),

$$V_{L2}=V_{L1}+A_cT$$

and evaluating,

$$V_{L2}=10+2\times30=10+60$$

$$=70 \text{ f.p.s.} \quad \textit{Ans.}$$

Classification of Motion. Motion can be classified as follows:

$$\text{(a) Plane Motion}\begin{cases}\text{Translation}\\ \text{Rotation}\end{cases}$$

(b) Helical Motion
(c) Spherical Motion

Plane Motion. When a body has a motion such that all points of that body trace paths which are in the same plane or in parallel planes, the body is said to have *Plane Motion.* If the paths of all points of the body are straight lines, the motion is called *Translation.* If the paths of all points are circles whose centers are either coincident or on the same straight line, the motion is called *Rotation.* The straight line upon which all the centers of the circular paths are located is called the *Axis of Rotation* of the body, and the planes of these circular paths are in every case perpendicular to this axis. The axis of rotation may pass through the body itself or it may lie entirely outside the body. The motion of a piston in its cylinder is an example of translation, while the motion of a pulley, flywheel, or gear is an example of rotation.

Plane motion of either translation or rotation is by far the most common and the simplest type of motion. It lends itself nicely to the use of skeleton diagrams of mechanisms in the graphical solutions of problems, for the lines of the links can be taken in projection on a plane parallel to the plane of motion. The paths of all points will project to this plane in their true shapes and the relative motion of each point of the mechanism can be studied. When motion of rotation is involved, an axis of rotation will project to the plane of the diagram as a point. This point is generally spoken of as the center of rotation even though in reality it is the axis of rotation.

Helical Motion. A body is said to have *Regular Helical Motion* when every point of the body moves in such a manner that it traces a path which is a helix.

A projection drawing of a helix is shown in Fig. 12. This figure shows that a helix is a curve formed by a point which, while maintaining a fixed distance from an axis, moves uniformly around that axis and parallel to it at the same time. Since the generating point maintains a fixed distance from the axis, its path lies in a cylindrical surface as shown in Fig. 12. Therefore the lower or plan view of the helix is coincident with the circumference of the circle which represents the plan view of the cylinder.

The upper view or front elevation of the helix shows the latter passing back and forth across the projection of the cylinder until, in one revolution, it has progressed upward along the cylinder a distance equal to its so-called *Lead.* In order to make this drawing, the circular plan view (which represents one revolution around the axis of the cylinder) is here divided into sixteen equal parts (arcs). Then each division represents one-sixteenth of a revolution. The distance the generating point is to advance along the axis in one revolution is divided therefore into sixteen equal divisions as shown; for, in one-sixteenth of a revolution about the axis, the generating point must advance upward along, or parallel to, the axis a distance equal to one-sixteenth of the lead. Therefore a vertical line drawn upward from any point, such as point *1* on the circle, intersects the horizontal line drawn through point *1* on the other view, and in this manner the various points on the front elevation are determined.

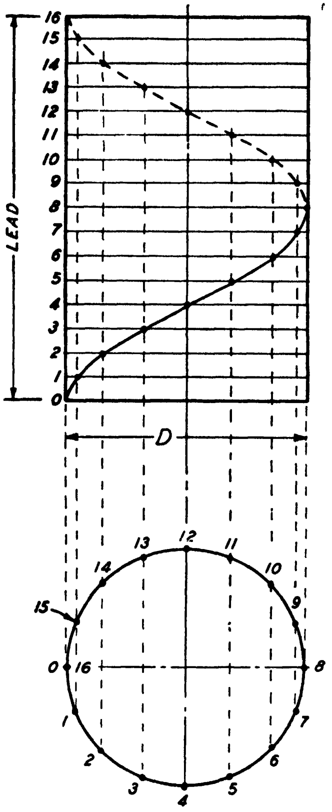


Fig. 12. Helix

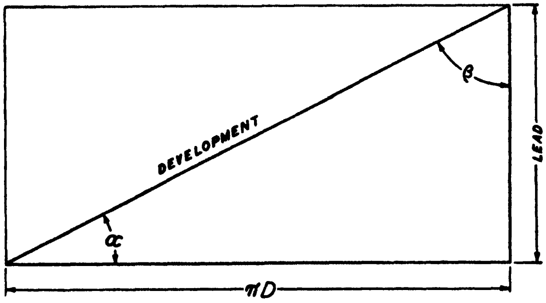


Fig. 13. Development of Fig. 12 (not to scale)

An example of helical motion is the nut and bolt. When a nut is screwed onto a bolt, every point of the nut traces a path which is a regular helix.

The Helix and its Angles. A cylinder has a development or pattern which is a rectangle. The base of the rectangle is equal in length to the circumference of the base of the cylinder. The height of the rectangle is equal to the altitude of the cylinder. The rectangle of Fig. 13 is the development of the cylinder of Fig. 12 in which the height of the cylinder is the lead of the helix drawn upon it. Now if the diagonal of this development is drawn, it becomes the true development of the helix, and the complementary angles α and β (alpha and beta), as shown in Fig. 13, are the so-called *Angles of the Helix.* Angle α is the angle that any tangent to the helix makes with the plane of the base of the cylinder. Angle β is the angle that any tangent makes with that element of the cylinder which is drawn through the point of tangency.

From the figure, it is evident that

$$\tan \alpha = \frac{L}{\pi D} \tag{6}$$

or

$$\alpha = \tan^{-1}\frac{L}{\pi D} \tag{7}$$

and

$$\tan \beta = \frac{\pi D}{L} \tag{8}$$

or

$$\beta = \tan^{-1}\frac{\pi D}{L} \tag{9}$$

and

$$\alpha + \beta = 90^\circ \tag{10}$$

In the above formulas,

L = the lead of the helix in inches.

D = the diameter of the cylinder in inches.

Spherical Motion. When the motion of a body is such that the path of every point of the body lies in some spherical surface, the body is said to have Spherical Motion. Those types of fly-ball governors, in which the balls rotate in different planes as the speed changes, are examples of spherical motion.

Since both spherical and helical motion are of infrequent occurrence in Mechanism, *hereafter any mention of motion in this volume refers to plane motion, unless otherwise stated.*

Continuous Motion. When a point continues indefinitely to move along its path without reversing its direction, the motion of the point is said to be *Continuous,* and its path must be a closed curve. The curve most frequently used is the circle.

Intermittent Motion. This type of motion is that in which a point is caused to rest or dwell at certain intervals, after which it proceeds to move along its path in the same direction.

Reciprocation. When a point moves forward and backward, retracing its path upon arriving at either end thereof, the point has a motion of *Reciprocation.* This term is generally restricted in use to those cases in which the path is a straight line. This is sometimes called *rectilinear reciprocation.*

Oscillation. A reciprocating motion in which the path is a circle (or an arc of a circle) is called *Oscillation.* It is often referred to as *rotative reciprocation.*

Cycle, Period, and Phase. At the instant of starting its motion, the links of a mechanism have a certain relative position or placement with respect to each other to which they are returned repeatedly by a series of movements. Such a series of movements is called a *Cycle.* The four-cycle gas engine is so named because the piston must make four strokes before all the links of the mechanism have completed such a series of movements.

The time which is necessarily used in completing a single cycle is referred to as the *Period.*

The relative positions which are taken by the links of a mechanism at any instant of the cycle is spoken of as a *Phase.*

Linear Velocity of a Point in Rotation. The speed of rotation of a body is generally described by giving the number of turns per minute which the body makes about its axis. This is spoken of as its number of revolutions per minute or as its r.p.m. Evidently the r.p.m. of any point of the body is the same as the r.p.m. of the body itself.

The linear displacement per unit of time, or the linear velocity, V_L, of a point of a rotating body is dependent on its r.p.m. In Fig. 14, c is a point on the rim of pulley A which rotates about its axis or center, O, with an r.p.m., N. R_1 is the radius of the pulley in feet, and hence is the distance from c to the center of rotation, O. In one revolution of A, point C will travel a distance which is equal to the circumference of its circular path. This distance is equal then to $2\pi R_1$ feet. In N revolutions it will travel N times as far as in one revolution, or a distance equal to $2\pi R_1 N$ feet. Since the pulley makes N revolutions per minute, this distance is traveled by point c in one minute. Therefore V_{L1} (the linear velocity of c) is equal to $2\pi R_1 N$ f.p.m. Evidently the linear velocity V_{L2} (of point b on the hub) will equal $2\pi R_2 N$ f.p.m., if R_2 is also given in feet. Or, in general, for any point of a rotating body,

$$V_L = 2\pi RN = \pi DN \text{ f.p.m.} \tag{11}$$

in which

V_L = the linear velocity in f.p.m.
R = the radius of the path of the point in feet
$D = 2R$ = the diameter of the path of the point in feet
N = the r.p.m. of the point.

Note. From the foregoing it should be evident that the unit of the linear velocity, V_L, will always have as its linear part the unit which is used for R, and for its time part the unit used for N. Therefore, if R were given in inches with N in revolutions per minute, and these values substituted as such in formula (**11**), the unit of V_L would be inches per minute, which of course could then be changed to feet per minute by dividing by 12.

It is now necessary to derive a relationship between the linear velocities of any two points of the same rotating body. Considering points b and c as any two points of pulley A of Fig. 14, then,

$$V_{L1} = 2\pi R_1 N$$
$$V_{L2} = 2\pi R_2 N$$

Dividing the former by the latter,

$$\frac{V_{L1}}{V_{L2}} = \frac{2\pi R_1 N}{2\pi R_2 N}$$

Canceling the common factors in the second member,

$$\frac{V_{L1}}{V_{L2}} = \frac{R_1}{R_2} \tag{12}$$

It is very important for the student to note that this formula states that the linear velocities of any two points of the same rotating body are directly proportional to their radii. In formula (**12**) it is necessary to have R_1 and R_2 in the same linear unit, but what unit is used is of no consequence.

Example 1. A pulley is 40 inches in diameter and makes 180 revolutions per minute. Required, the linear velocity in feet per minute of a point located on the pulley rim.

Solution. Here $R=\frac{40}{2}=20$ in. or $\frac{20}{12}$ ft. and $N=180$ r.p.m. Substituting these values in formula (11),

$$\begin{aligned} V_L &= 2\pi RN \\ &= 2\times\pi\times\tfrac{20}{12}\times180=600\pi \text{ f.p.m.} \\ &= 600\times3.1416=1885- \text{ f.p.m.} \qquad \textit{Ans.} \end{aligned}$$

Example 2. If the hub of the pulley of the preceding example has a diameter of 5 inches, what is the linear velocity in feet per minute of a point located on the hub, as point *b* of Fig. 14?

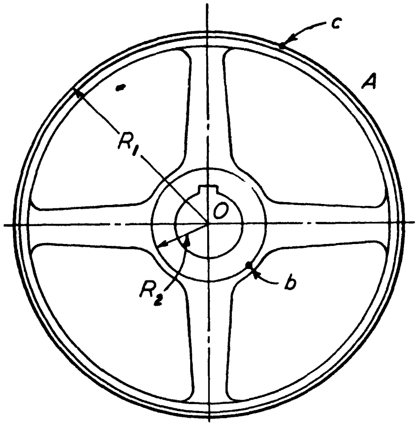


Fig 14

Solution.

First Method: Here $R=\frac{5}{2}$ in. or $\frac{\frac{5}{2}}{12}$ ft., and $N=180$ r.p.m. Applying formula (11),

$$V_L=2\pi RN$$

and evaluating therein,

$$\begin{aligned} V_L &= 2\times\pi\times\frac{\frac{5}{2}}{12}\times180 \\ &= 2\times\pi\times\tfrac{5}{24}\times180=75\pi \\ &= 235.6 \text{ f.p.m.} \qquad \textit{Ans.} \end{aligned}$$

Second Method: Applying formula (12),

$$\frac{V_{L1}}{V_{L2}}=\frac{R_1}{R_2} \text{ or } \frac{V_{L2}}{V_{L1}}=\frac{R_2}{R_1}$$

and evaluating therein with $V_{L1}=1{,}885$ f.p.m., $R_1=20$ in. and $R_2=\frac{5}{2}$ in.,

$$\frac{V_{L2}}{1885}=\frac{\frac{5}{2}}{20}=\frac{5}{40}=\frac{1}{8}$$

Multiplying both members of the equation by 1,885,

$$V_{L2}=\tfrac{1}{8}\times 1885=235.6 \text{ f.p.m.} \qquad \textit{Ans.}$$

The Radian. Before proceeding any further with our discussion of velocity, it is necessary to introduce a unit of angular measure, called the radian. It is a much larger unit of angular measure than the degree. Hence if an angle is measured using the radian as a unit, it will be represented by a much smaller number than when it is measured using the degree as a unit.

A central angle is an angle whose vertex is at the center of a circle. Angle θ (theta) of Fig. 15 is such an angle. It will be seen from the figure that its sides are necessarily radii of the circle. Now the sides of any central angle cut off or intercept an arc on the circumference of the circle. When a central angle is an angle of one radian, as the angle θ of Fig. 15, its radial sides must intercept an arc equal in length to the radius of the circle, no matter what size of circle is used. Hence, in the figure, the length of arc MN is equal to the radius R. It is evident that there can be as many angles of one radian each, or as many radians, at the center of a circle as there are arcs of a length equal to the radius in the circumference of the circle. In other words, if the circumference is divided by R, the radius of the circle, the result will be the number of arcs of this length in the circumference, and hence the number of radians in the whole central angle of 360 degrees at the center of the circle. By performing this division and obtaining this relationship between

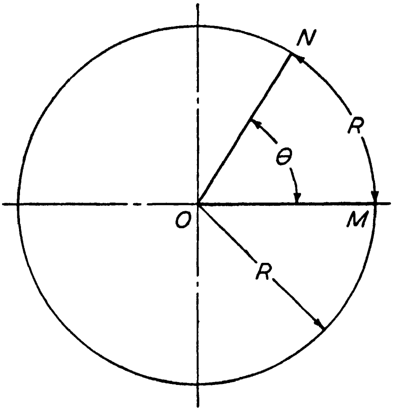


Fig. 15. An Angle, θ, of 1 Radian

the radian and the degree, an angle given in one unit may be changed into the other unit. Thus since the circumference of a circle is equal to $2\pi R$, dividing by R gives

$$\frac{2\pi R}{R}=2\pi$$

the number of radians in 360 degrees, so that

$$2\pi \textit{ radians}=360 \textit{ degrees} \tag{a}$$

Dividing step (*a*) by 2π

$$1 \textit{ radian}=\frac{360}{2\pi} \textit{ degrees} \tag{b}$$

Dividing step (*a*) by 360,

$$1 \textit{ degree}=\frac{2\pi}{360} \textit{ radians} \tag{c}$$

From step (*b*) it is evident that an angle given in radians can be changed into degrees by multiplying by $\frac{360}{2\pi}$. From step (*c*) it is evident that an angle given in degrees can be changed into radians by multiplying by $\frac{2\pi}{360}$.

In connection with the radian it is interesting to note that if a central angle, θ, is given in radians, the arc, S_a, intercepted by its radial sides, can be found by the formula

$$S_a=R\theta \tag{13}$$

In this formula, S_a must be given in the same linear unit as R, the radius.

Example 1. It is required to express an angle of 160 degrees using the radian as a unit.

Solution.

$$160°=160\times 1°$$

From step (*c*), $1°=\frac{2\pi}{360}$ radians; hence, substituting for 1° its equal, $\frac{2\pi}{360}$ radians,

$$160°=160\times\frac{2\pi}{360}=2.79 \text{ radians.} \qquad \textit{Ans.}$$

Example 2. It is required to express an angle of 2 radians using the degree as a unit.

Solution. 2 radians $=2\times 1$ radian.

From step (*b*) 1 radian $=\frac{360}{2\pi}$ degrees.

Therefore 2 radians $=2\times\frac{360}{2\pi}=114.6°$. *Ans.*

Angular Velocity. It has been shown that a point which rotates about a center has a linear velocity or a linear displacement along its circular path per unit of time. Such a point also sweeps through or displaces a central angle at the center of its circular path and hence has an angular displacement per unit of time. This angular displacement per unit of time is known as the *Angular Velocity* of the point and will be designated by the symbol V_a. Hence any point having a motion of rotation has not only a linear velocity but an angular velocity.

The r.p.m. of a point, or of the rotating body to which the point belongs, has been introduced already in this chapter. It is in one sense an angular velocity because one revolution or turn is surely an angular displacement about the center of rotation. The r.p.m. of a point has a great practical significance as a speed of rotation, but in analytical work the angular velocity in radians per unit of time is necessarily introduced. The expression *Angular Velocity* therefore specifies a rotative speed in radians per unit of time.

Let us consider point *c* of the rotating body, *A*, of Fig. 14. In one revolution or turn about the center *O*, point *c* will sweep through or displace a central angle of 360 degrees, or 2π radians. In two turns, an angle two times as large would be displaced, and in *N* turns an angle *N* times as large would be displaced. This would be an angle of $2\pi N$ radians. Now if these *N* turns are made in one minute, or, in other words, if *N* is the r.p.m. of *A* and hence of all of its points such as *c*, the central angle of $2\pi N$ radians will be displaced by point *c* in one minute. It therefore, by definition, becomes the angular velocity of point *c*.

It will be noted that any other point of the rotating body *A* would have given the same result. Hence the angular velocities of all points of the same rotating body are equal, and this angular velocity which they all possess may be considered as the angular velocity of the body as well, and is given by the formula

$$V_a=2\pi N \text{ radians per minute} \qquad (14)$$

Note. An inspection of this formula shows that V_a is a function of N only. Since N must be the same for all points of the same rotating body, V_a as stated before must be the angular velocity of the body as well as of every point contained therein. It should also be noted in formula (**14**) that if the number of revolutions per second (r.p.s.) is introduced in the place of the r.p.m., V_a will be in radians per second.

Example 1. Find the angular velocity of a flywheel in both radians per minute and radians per second. The flywheel has a diameter of 5 feet and makes 250 revolutions per minute.

Solution. Here $N=250$ r.p.m. Applying formula (**14**),

$$\begin{aligned} V_a &= 2\pi N \\ &= 2\pi \times 250 = 500\pi \\ &= 1570.8 \text{ radians per min.} \qquad \textit{Ans.} \end{aligned}$$

$$\frac{1570.8}{60} = 26.18 \text{ radians per sec.} \qquad \textit{Ans.}$$

Example 2. What is the speed in r.p.m. of a rotating body whose angular velocity is equal to 15 radians per second?

Solution.

15 radians per second $=15\times60$ radians per min.; therefore $V_a=15\times60=900$ radians per min. Substituting this value of V_a in formula (**14**),

$$V_a = 2\pi N$$

$$900 = 2\pi N$$

Dividing both members of this equation by 2π,

$$N = \frac{900}{2\pi} = 143.2 \text{ r.p.m.} \qquad \textit{Ans.}$$

Relationship between Linear and Angular Velocities. Since a point in rotation has both a linear velocity and an angular velocity, there must be some relationship between those two velocities.

From formula (**11**) the linear velocity of a point is

$$V_L = 2\pi RN$$

and from formula (**14**) the angular velocity is

$$V_a = 2\pi N$$

There is an axiom in mathematics which states that if equals are divided by equals, their quotients are equal. Hence if V_L is divided by V_a, and $2\pi RN$ by $2\pi N$, the quotients thus produced are equal to each other, or

$$\frac{V_L}{V_a} = \frac{2\pi RN}{2\pi N}$$

Canceling the like factors in the numerator and denominator of the second member of this equation,

$$\frac{V_L}{V_a}=R \tag{15}$$

This formula states that the ratio existing between the linear and angular velocities of the same point is always equal to the radius of the point, which is the distance from the point to its center of rotation.

Multiplying both members of formula (**15**) by V_a,

$$V_L=RV_a \tag{16}$$

Formula (**16**) states that the linear velocity of a point is equal to the product of its radius and angular velocity.

Dividing both members of formula (**16**) by R,

$$V_a=\frac{V_L}{R} \tag{17}$$

Formula (**17**) states that the angular velocity of a point is equal to its linear velocity divided by its radius.

In applying formulas (**15**), (**16**), and (**17**), care must be exercised in the use of correct units for the various factors, V_L, V_a, and R. The truth of the following statements should be recognized:

(*a*) The time unit involved in V_L and V_a must always be the same.

(*b*) The space (distance) unit involved in V_L and R must always be the same.

(*c*) The angular unit involved in V_a is always the radian.

Example 1. The linear velocity of a point on a 28-inch pulley is 3,500 feet per minute. Find:

(*a*) the angular velocity of the point (or pulley) in radians per minute and in radians per second.

(*b*) the r.p.m. of the pulley.

Solution. (*a*) Here $V_L=3{,}500$ f.p.m. and $D=28$ in.

$$R=\frac{D}{2}=\frac{28}{2}=14 \text{ in. or } \tfrac{14}{12} \text{ ft.}$$

Substituting these values in formula (**17**),

$$V_a=\frac{V_L}{R}$$

$$V_a=\frac{3500}{\frac{14}{12}}=3500\times\tfrac{12}{14}$$

$$=3000 \text{ radians per min.} \quad \textit{Ans.}$$

$$\therefore\ V_a=\frac{3000}{60}=50 \text{ radians per sec.} \quad \textit{Ans.}$$

(**b**) First Method: Substituting the value of V_a in radians per minute in formula (**14**),

$$V_a=2\pi N$$

$$3000=2\pi N$$

Dividing both members of the equation by 2π,

$$N=\frac{3000}{2\pi}=477.5 \text{ r.p.m.} \quad \textit{Ans.}$$

(**b**) Second Method: Substituting the values of V_L in f.p.m. and D in ft. in formula (**11**),

$$V_L=\pi DN$$

$$3500=\pi\times\tfrac{28}{12}\times N$$

Multiplying by 12,

$$12\times3500=\pi\times28\times N$$

Dividing by $\pi\times28$,

$$N=\frac{12\times3500}{\pi\times28}=477.5 \text{ r.p.m., to check ans.}$$

Example 2. A pulley which is 4 feet in diameter rotates at 80 revolutions per minute.

(*a*) Find the linear velocity in feet per minute of a point on the rim of the pulley, using formula (**11**).

(*b*) Find the angular velocity of the same point in radians per minute, using formula (**14**).

(*c*) Check the value of V_L as found in part (*a*) by using formula (**16**) with the value of V_a as found in part (*b*).

Solution. (*a*) Here $D=4$ ft., and $N=80$ r.p.m. Applying formula (**11**),

$$V_L=\pi DN$$

and evaluating therein,

$$V_L=\pi\times4\times80=1005.3 \text{ f.p.m.} \quad \textit{Ans.}$$

(*b*) Substituting the known value of N in formula (**14**),

$$V_a=2\pi N$$

$$=2\pi\times80=502.66 \text{ radians per min.} \quad \textit{Ans.}$$

(*c*) Using formula (16) with $R=2$ ft. and $V_a=502.66$ radians per min.,

$$V_L=RV_a$$
$$=2\times502.66=1005.3 \text{ f.p.m. to check ans.}$$

Graphic Representation of Motion. In the graphic solutions of kinematic problems, the velocity and direction of motion of a point may be represented by a straight line. The numerical value or magnitude of the velocity is represented by, and hence determines, the length of the line; while the direction of motion is represented by the direction of the line. An arrow is placed at the end of the line to further define the direction of motion. If the path of the point is known, the direction of motion is the direction of the path, as was brought out earlier in this chapter when dealing with Figs. 10 and 11.

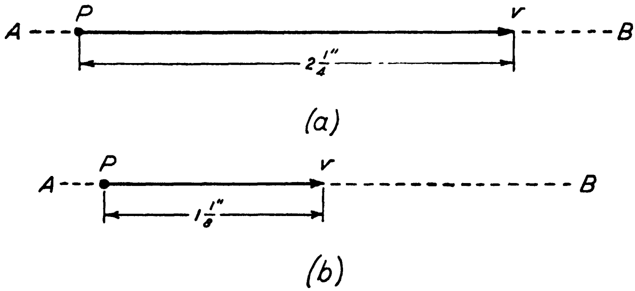


Fig. 16

In Fig. 16(*a*), point *P* is moving along the straight-line path *AB*. It has a velocity of 225 feet per minute at the instant shown by its position on its path. Since the path is a straight line, the direction of the line *Pv*, which is to represent the motion of *P*, will be along its path. Now in order to secure a length of *Pv* which will represent the magnitude or numerical value of the velocity, some so-called scale must be adopted. In this figure, a scale of 100 feet per minute to the inch (or 1 in. = 100 f.p.m.) was selected. This means that there will be one inch of line to represent each 100 feet per minute of the velocity, so that the length of the line in inches will be the quotient obtained by dividing the velocity by the scale, or $\frac{V_L}{scale}$ inches. In this case,

$$\frac{V_L}{scale}=\frac{225}{100}=2\tfrac{1}{4}\text{ inches}$$

Hence *Pv* of Fig. 16(*a*) is 2¼ inches in length. Such a line (*Pv*) is called a *Vector.*

Point *P*, with the same velocity and path, is again shown in Fig. 16(*b*), but in this case a scale of 200 feet per minute to the inch was adopted. So that the length of *Pv* is here equal to

$$\frac{V_L}{scale}=\frac{225}{200}=1\tfrac{1}{8}\text{ inches}$$

In Fig. 17, point *P* is rotating about the center, *O*, with a velocity of 15 feet per second. Having given the position of the point and its center of rotation, its circular path was described with the distance *OP* as a radius. The tangent to the path was then drawn. Having selected a scale of 20 feet per second to the inch, the length of the vector *Pv* was found to be

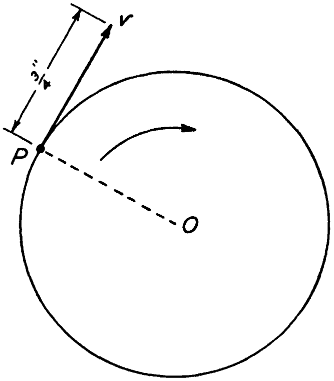


Fig 17

$$\frac{V_L}{scale}=\frac{15}{20}=\tfrac{3}{4}\text{ inch}$$

This length was then laid out on the tangent in the direction that was indicated by the direction of rotation.

Example 1. Find the length of the vector to represent the motion of point *P* of Fig. 16 when the scale is 1 inch = 75 feet per minute.

Solution. Here $V_L=225$ f.p.m., so that

$$\frac{V_L}{scale}=\frac{225}{75}=3\text{ in.}\quad \textit{Ans.}$$

Example 2. Find the length of the vector to represent the motion of point P of Fig. 17 when the scale is 1 inch = 10 feet per second.

Solution. Here $V_L = 15$ f.p.s.

$$\frac{V_L}{scale} = \frac{15}{10} = 1\tfrac{1}{2} \text{ in.} \quad \textit{Ans.}$$

Composition of Motions or Velocities. The motion of a body may be due to the action of a single force, or it may be due to the combined action of two or more forces. In the case of the single force, the body will be caused to move in a straight line whose direction is the same as that of the force. In the case of two or more forces it is evident that the body cannot move in the direction of each force, for it can have at any instant a motion along only one straight line. Hence the body must have a motion whose velocity and path are the result of the combined effects of the several forces acting upon it.

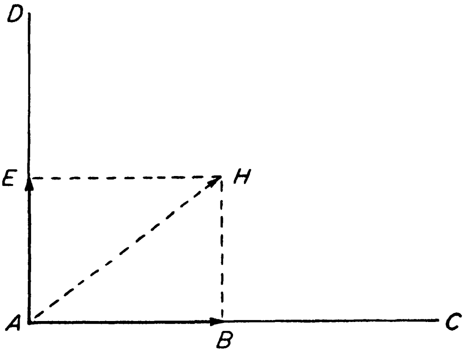


Fig. 18

The latter can be demonstrated by assuming that A of Fig. 18 represents a body which is lying on a table. Let us suppose that AC and AD are strings making a right angle with each other. First, let the string AC be pulled with such a force that it will cause the body, A, to move one inch toward C in one second; then A will be at B at the end of a second, the line AB being a vector representing the direction and velocity of A when the string AC is pulled. Next, suppose the body to be back in its initial position A, and that the string AD is pulled with such a force that it will cause the body to move ¾ inch toward D in one second; then at the end of a second, *A* will be at *E*, the line or vector *AE* representing the direction and velocity of the body when the string *AD* is pulled.

Now suppose both strings to be pulled at the same time, each being pulled by the same force as before. Then the pull on *AC* will still cause *A* to move toward *C* with a velocity *AB*, and the pull on *AD* will cause *A* to move toward *D* with a velocity *AE;* so that, at the end of a second, the body will be at neither *E* nor *B*, but at some point, *H*, whose position is at a distance *BH* from *B* (the line *BH* being equal and parallel to *AE*). The point *H* is also at a distance *EH* from *E* (the line *EH* being equal and parallel to *AB*). In other words, the path over which the body has actually moved, under the simultaneous action of the combined forces, is the line *AH*. The latter is the diagonal of a parallelogram whose sides are equal to *AB* and *AE*. The lines *AB* and *AE*, which represent the velocities caused by the pulls on the respective strings, must be drawn to the same scale; that is, if the pull on *AC* causes a velocity of one inch per second and the pull on *AD* causes a velocity of ¾ inch per second, *AE* must be ¾ as long as *AB*.

The velocities *AB* and *AE* are called *Component Velocities* and the velocity *AH* is called the *Resultant Velocity*. The resultant vector, *AH*, can now be measured. The length thus obtained, if multiplied by the scale used in the drawing will give the magnitude or numerical value of the resultant (or actual) velocity of body *A;* and the direction of *AH* is the actual direction that the body will take when under the influence of a system of two forces which tend, individually, to give the body the velocities that are used along *AC* and *AD*.

The parallelogram of Fig. 18 is called a *Parallelogram of Velocities*. The figure shows that if vectors representing the component velocities of a point (or body) are drawn to some scale and used as the sides of a parallelogram, the diagonal of the parallelogram thus created will be the resultant velocity of the point to the same scale. The parallelogram need not always be a rectangle; that is, the component velocities such as *AB* and *AE* of Fig. 18 may be at any angle with each other. However the right angle condition is that which will be applied in this volume.

Taking another example: suppose a ball is thrown toward the east with a velocity of 10 feet per second, and the wind, acting as a second force, carries the ball toward the south with a velocity of 5 feet per second. Let us find, graphically, how fast and in what direction the ball is actually moving. Using a velocity scale of 1 inch = 4 feet per second, the velocity toward the east will be represented by a vector 2½ inches in length; for

$$\frac{V_L}{scale}=\frac{10}{4}=2\tfrac{1}{2}\text{ inches}$$

The velocity toward the south will be represented by a vector, 1¼ inches in length, for

$$\frac{V_L}{scale}=\frac{5}{4}=1\tfrac{1}{4}\text{ inches}$$

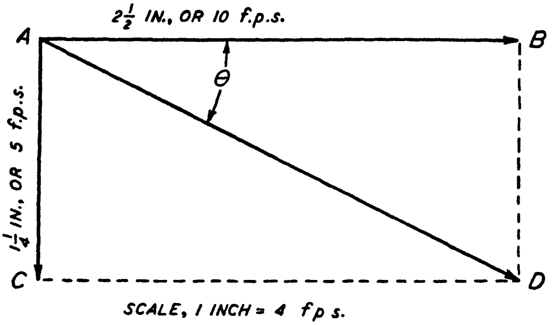


Fig. 19

Selecting *A* as the initial position of the ball (see Fig. 19) first draw line *AB* to the right (east) and equal in length to 2½ inches; then draw *AC* vertically downward (south) and equal in length to 1¼ inches. Through point *B* draw line *BD* parallel to *AC*, and through point *C* draw *CD* parallel to *AB*. This gives the parallelogram, the diagonal *AD* of which is the vector representing the resultant velocity, furnishing the magnitude of the velocity and the direction of motion of the ball. Measuring *AD*, it is found to be 2.8 inches in length. The actual velocity of the ball is obtained by multiplying this length by the velocity scale employed. Thus

$$2.8\times 4=11.2\text{ feet per second. } \textit{Ans.}$$

The direction of the motion of the ball can be recorded by the angle which *AD* makes with either the horizontal or vertical. Measuring the former, θ of Fig. 19, by using a protractor, it is found to be 26½ degrees or 26 degrees 30 minutes (26°30′). *Ans.*

The preceding answers can be checked mathematically as follows:

$$AD = \sqrt{(AB)^2 + (AC)^2}$$

for the hypotenuse of a right triangle is equal to the square root of the sum of the squares of the other two sides. Therefore

$$AD = \sqrt{10^2 + 5^2} = \sqrt{125} = 11.2- \text{ f.p.s.} \quad \textit{Ans.}$$

and

$$\begin{aligned} \theta &= \tan^{-1}\frac{BD}{AB} = \tan^{-1}\frac{AC^{*}}{AB} \\ &= \tan^{-1}\tfrac{5}{10} \\ &= 26°34' \end{aligned}$$

which is a close check to the result as found with the protractor.

The process which has been used in the solution of these examples is known as the *Composition of Velocities.*

Resolution of Velocities. Frequently, in the subject of Mechanism, the real direction and velocity of a body (or point) are given, with which to find the components of this velocity that are parallel and at right angles to some given line. Thus suppose a point, A, is moving at a velocity of 100 feet per minute and in a direction along the given line AC of Fig. 20, and its speed must be determined at this instant in directions parallel to (or along) and at right angles to the given line MN which goes through point A.

To obtain a graphical solution of this problem, the given lines MN and AC are drawn. Having selected a convenient scale of 50 feet per minute to the inch, the length of the vector to represent the velocity of point A is obtained by dividing the velocity, 100 feet per minute, by the scale. This gives a length of 2 inches for the vector, which is then laid off from A on line AC, giving the vector AD. Now through points A and D, lines AB and DB are drawn, respectively perpendicular and parallel to MN, and intersecting at B. AB is then the component of the velocity of A (and hence is the velocity of A) in a direction at right angles to MN, and BD is the component of the velocity of A (and hence is the velocity of A) in a direction parallel to or along MN. Since these components are drawn to the same scale as the vector AD, their lengths are measured and converted into feet per minute by multiplying them by that scale. Thus

---

*$\tan^{-1}\frac{BD}{AB}$ = the angle whose tangent is $\frac{BD}{AB}$. It is read "antitangent $\frac{BD}{AB}$" or "arc tangent $\frac{BD}{AB}$." Therefore any anti- or arc trigonometric function is an angle.

Length of $AB = 1.73$ inches

∴ Component $AB = 1.73 \times 50 = 86.5$ f.p.m. *Ans.*

Length of $BD = 1$ inch

∴ Component $BD = 1 \times 50 = 50$ f.p.m. *Ans.*

The preceding process, by which the components of a given velocity are found, is called *Resolution of Velocities.*

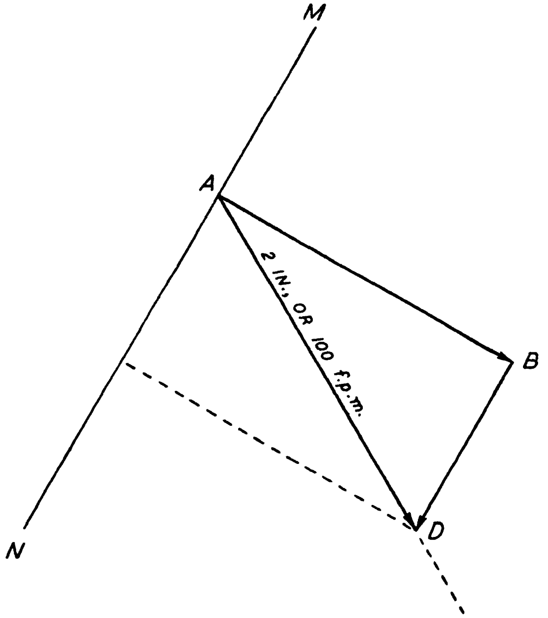


Fig. 20

Methods of Transmitting Motion. Mechanisms, as stated heretofore, are designed to modify and transmit motion. Various elementary mechanisms, which deal directly with such modification and transmission and which when grouped together form the more complex mechanisms, are introduced in succeeding chapters, and the fundamental principles and laws governing their action are studied. Although the number of forms of these elementary mechanisms is practically without limit, they can be classified, as follows, by the methods which they employ in the transmission of motion from driver to follower:

1. Direct contact

2. Intermediate rigid link connector
3. Intermediate flexible link connector

A *Direct-Contact Mechanism,* as the name implies, is one in which the acting surfaces of driver and follower are in contact so that motion is imparted directly from driver to follower. Examples of such mechanisms are as follows: gears, cams, cylindrical frictions, conical or bevel frictions, etc.

A mechanism that employs a rigid link, through which motion from the driver is imparted to the follower, is known as a *Rigid-Link Connector Mechanism.* Such a mechanism is often referred to simply as a Linkage. The rigid link, which is placed between the driver and the follower, is spoken of as a *Connector* or *Connecting Rod.* Typical examples of this type of mechanism are the four-bar linkage of Fig. 8 and the slider-crank mechanism of Fig. 9.

If the connector which accepts the motion from the driver and imparts it to the follower is a flexible link, the mechanism is called a *Flexible-Link Connector Mechanism* or a *Wrapping-Connector Mechanism.* Of course such a link can be subjected to a pull only, whereas a rigid intermediate connector can be subjected to either a pull or a push (thrust). Wrapping-connector mechanisms include all those mechanisms which employ belts, bands, ropes, or chains between the driver and its follower. (See Figs. 66, 74, 86, and 88.)

QUESTIONS AND PROBLEMS

1. A train travels a distance of 450 miles in 7½ hours. Find its velocity or speed in miles per hour (m.p.h.). *Ans.* 60 m.p.h.

2. Find the velocity of the train of the preceding problem in miles per minute (m.p.m.), feet per minute (f.p.m.), and feet per second (f.p.s.). *Ans.* 1 m.p.m.; 5,280 f.p.m.; 88 f.p.s.

3. Change a velocity of 30 inches per second into velocity in feet per minute. *Ans.* 150 f.p.m.

4. If a trip of 2,500 miles is made by automobile at an average speed of 42 miles per hour, find the time consumed in making the trip. *Ans.* 59.5+ hr.

5. An engine with a 5-inch stroke is running at a speed of 2,500 revolutions per minute. Find the mean piston velocity in feet per minute. *Ans.* 2,083⅓ f.p.m.

6. Find the acceleration in feet per minute per minute, in feet per minute per second, and in feet per second per second, of a point on the rim of a flywheel, if the velocity of the point changes from 1,800 feet per minute to 3,000 feet per minute in 1½ minutes. *Ans.* 800 ft. per min. per min.; 13⅓ ft. per min. per sec.; ²⁄₉ ft. per sec. per sec.

7. Classify *motion.*

8. How does *angular velocity* differ from *linear velocity?*

9. Define *the radian.*

10. A pulley is 30 inches in diameter and rotates at a speed of 240 revolutions per minute. Find the linear velocity in feet per minute of a point on the rim of the pulley. *Ans.* 1885 f.p.m.

11. Find the linear velocity in feet per minute of a point on an arm of the pulley of the preceding problem if the point is located 9 inches out from the axis of the pulley. *Ans.* 1131 f.p.m.

12. A point on the rim of a 5-foot flywheel has a linear velocity of 4712 feet per minute. What is the rotative speed of the flywheel in revolutions per minute? *Ans.* 300 r.p.m. nearly.

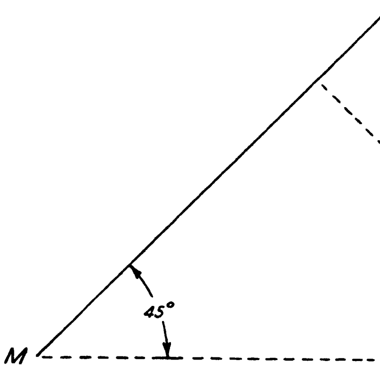


Fig 21

13. How many radians are there in a central angle of 165 degrees? *Ans.* 2.88 radians.

14. How many degrees are there in a central angle of $\frac{\pi}{2}$ radians? *Ans.* 90°.

15. A disk which is 8 inches in diameter rotates about an axis through its geometric center at a speed of 1000 revolutions per minute. Find:

(*a*) the linear velocity in feet per minute of a point on the rim of the disk

(*b*) the angular velocity in radians per minute and also in radians per second of the same point

(*c*) the angular velocity of the disk in radians per minute.

Ans. (*a*) 2,094.4 f.p.m. (*b*) 6,283.2 radians per min.; 104.7 radians per sec. (*c*) 6,283.2 radians per min.

16. The rim of a pulley has a velocity of 4,000 feet per minute. If the diameter of the pulley is 18 inches, find

(*a*) the r.p.m. of the pulley

(*b*) the angular velocity of the pulley in radians per second.

Ans. (*a*) 848.8 r.p.m. (*b*) 88.9 radians per second.

17. Define the term ***helix.***

18. What is the lead of a helix?

19. A helix with a lead of 1½ inches is generated on the surface of a cylinder whose diameter is 2 inches. Find

(*a*) the length of the development of the helix

(*b*) the angle α of the helix.

(*c*) the angle β of the helix. (See Fig. 13)

Ans. (*a*) 6.46 in. (*b*) 13°26′ (*c*) 76°34′

20. What length of vector should be used to graphically represent a velocity of 300 feet per minute if

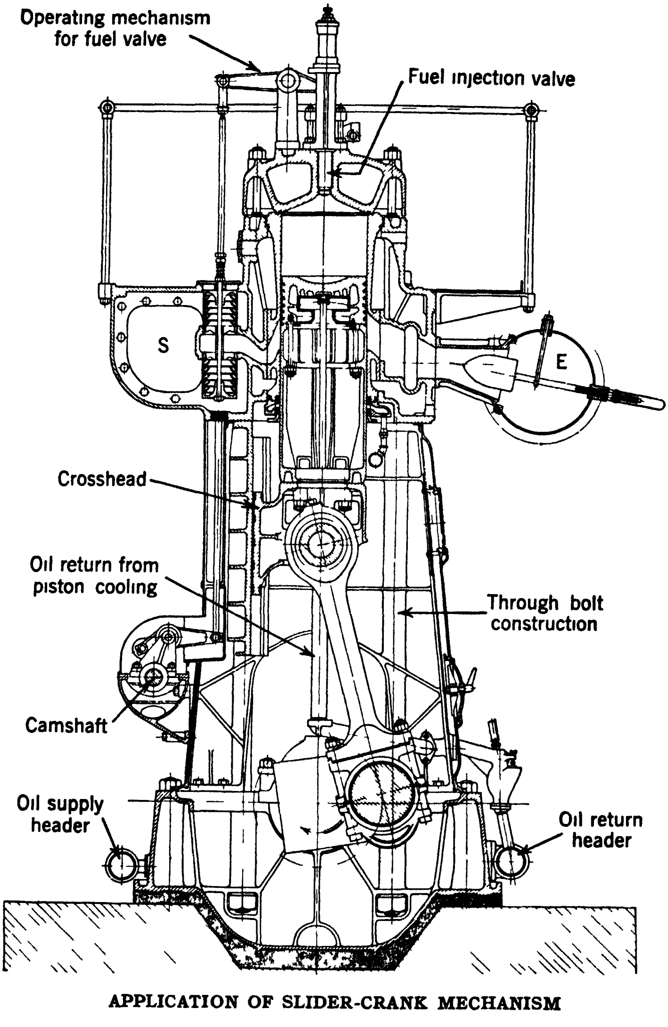
(*a*) a scale of 1 inch = 150 feet per minute is used,

(*b*) a scale of 1 inch = 400 feet per minute is used?

Ans. (*a*) 2 in. (*b*) ¾ in.

21. In Fig. 19, let the component velocities *AB* and *AC* of the point *A* be 15 feet per second and 6 feet per second respectively. Solve graphically for the magnitude of the resultant velocity, *AD*. *Ans.* 16.2 f.p.s.

22. In Fig. 21 assume point *A* as any point along the given line *MN*. Point *A* has a velocity of 300 feet per minute vertically downward along the line *AC* as shown. Required, to resolve the velocity of point *A* into components *AB* and *BD*, taken respectively perpendicular to and parallel to line *MN*. Measure the lengths of the component vectors and translate them into their magnitudes in feet per minute. Use a scale of 1 inch = 150 feet per minute. *Ans.* *AB* = *BD* = 1.41 inches. Therefore both components have a magnitude of 212 f.p.m. approximately.



Operating mechanism for fuel valve
Fuel injection valve
S
E
Crosshead
Oil return from piston cooling
Through bolt construction
Camshaft
Oil supply header
Oil return header

APPLICATION OF SLIDER-CRANK MECHANISM

CHAPTER III

INSTANT CENTERS AND INSTANTANEOUS MOTION

Uniform and Variable Motion. It has been seen in the preceding chapter that the motion of a point is along a tangent to its path. Hence, if that path is a curved line, the motion of the point is constantly changing in direction; while if the path is a straight line, the direction of motion of the point does not change (for a tangent to a straight line is coincident with that line). The velocity or speed of the point as it moves along its path may or may not be the same, as the point occupies its successive positions along that path. If the velocity is the same, the point is moving with a *Uniform Motion.* In such a case, the point is said to have a *Uniform* or *Constant Velocity.* If the velocity is not constant, the motion of the point is said to be *Variable.*

Definition of Instantaneous Motion. From the statements of the preceding article, it is evident that the motion of a point may or may not vary as to direction or velocity as the point takes or occupies its successive positions along its path. Since a moving point occupies one of its positions for only an instant, information as to its direction and velocity at that position, or at that instant, defines what is said to be the *Instantaneous Motion* of the point. Evidently the vector of point *A* of Fig. 20 represents the instantaneous velocity of that point; in other words, it defines the motion of the point for only that phase of its motion as given in the figure.

Instant Centers. As heretofore stated, all motion dealt with is relative motion. Thus in any mechanism each link has a certain relative motion at any instant with respect to each of the others. In Fig. 22, link *A* has a relative motion with respect to link *D,* and link *D* has the same relative motion with respect to *A;* that is, a pair of links have the same relative motion with respect to one another. So the number of different relative motions that can occur in a mechanism is equal to the number of different combinations of two each among its links. Then the relative motions that occur at any instant in the mechanism of Fig. 22 are as follows:

1. Relative motion of *A* with respect to *D* (or of *D* to *A*)
2. Relative motion of *B* with respect to *A* (or of *A* to *B*)
3. Relative motion of *C* with respect to *B* (or of *B* to *C*)
4. Relative motion of *C* with respect to *D* (or of *D* to *C*)
5. Relative motion of *A* with respect to *C* (or of *C* to *A*)
6. Relative motion of *B* with respect to *D* (or of *D* to *B*)

So in the motion of this mechanism there are six different relative motions occurring at any given instant, or six different relative instantaneous motions.

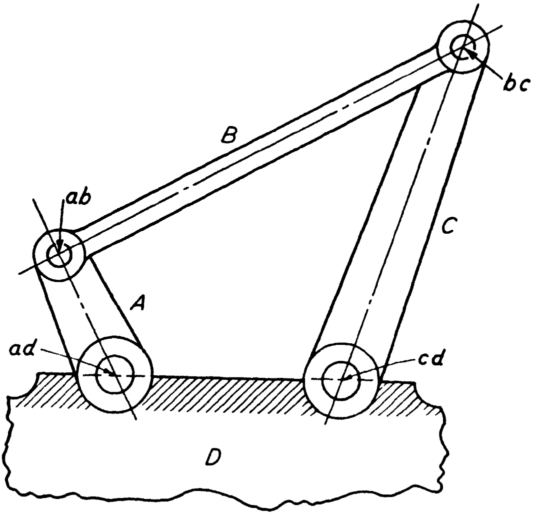


Fig 22

In an article that follows shortly, it will be proved that the instantaneous motion of one body with respect to another can be considered as one of rotation; that is, each body may be thought of as actually rotating with respect to the other. In that the relative rotation of any two bodies demands a center (axis) of rotation, there will be as many of these centers in any mechanism as there are different relative motions among the pairs of links, or six in the case of Fig. 22. And these centers of the instantaneous motion of a mechanism are known as *Instantaneous Centers* or, more briefly, *Instant Centers*. They are also commonly called *Centros*.

Designation of Instant Centers. In this text the instant center of the relative motion of any two bodies will be given or designated by the pair of lower case (small) letters which represent the two links involved. Thus in Fig. 22,

ad is the instant center of *A* with respect to *D*.
ab is the instant center of *A* with respect to *B*.
bc is the instant center of *B* with respect to *C*.
cd is the instant center of *C* with respect to *D*.

Permanent and Fixed Centers. Instant centers may be

1. Permanent centers
2. Fixed centers
3. Neither permanent nor fixed.

When two bodies are materially connected to each other in such a manner that one may turn or rotate with respect to the other (as is the case wherever a turning pair is used) their instant center (or axis) of rotation is called a *Permanent Center* (or permanent axis). Thus the center of any pin-joint connection is a permanent center; for example, *ad, ab, bc,* and *cd,* the centers of the four pin joints of Fig. 22.

When a turning pair is used to connect two bodies, one of which is the fixed link or frame of the mechanism, the instant center of the relative motion of the two bodies is called a *Fixed Center.* It is evident, then, that a fixed center is a permanent center which never changes its position with respect to the frame. Hence, of the four permanent centers of Fig. 22, those which are the centers of the pure rotation of links *A* and *C* with respect to the frame (namely, centers *ad* and *cd*) are fixed centers.

The six different relative motions of Fig. 22, however, require six instant centers. Of these six instant centers, only four are exposed in Fig. 22 and these have been seen to be permanent or fixed centers. The other two instant centers, *ac* and *bd,* are of the relative motions of *A* with respect to *C* and *B* with respect to *D.* These pairs of links, as seen in Fig. 22, have no direct physical connection to each other. The instant centers of their relative motions are of the third classification, being neither permanent nor fixed. They can not be found by inspection of the figure, hence a method must be developed by which they may be located.

Instantaneous Motion Assumed as Rotation. In Fig. 23, point *P* is shown moving along its path, *MM'*. The velocity and direction of its instantaneous motion, at the position shown, are given by the vector Pv which is drawn tangent to its path MM' at the given position. The line NN' is drawn perpendicular to tangent TT' at P. With O_1 located on NN' as a center, and O_1P as a radius, the circle 1-1′ is described. Another circle, 2-2′ is described with another point, O_2 on line NN', as its center, and the distance O_2P as a radius. Circles 1-1′ and 2-2′ then have, as their common tangent, line TT', which is likewise tangent to MM'. Since the vector Pv is coincident with TT', it is also tangent to the circles 1-1′ and 2-2′ at the point P. Therefore, at the given instant, the point P could be traveling along either circle, just as well as along its given path, and still have its motion defined by the vector Pv. It is also evident that the circles 1-1′ and 2-2′ are merely representative of an infinite number of circles which could be drawn with their centers somewhere along NN'. Then it follows that point P could be considered at the instant as traveling along any one of these circles. Its instantaneous motion would therefore be one of rotation and would be governed by all the laws of such a motion. The center of its instantaneous motion (its instantaneous center) can not be definitely fixed, but it will be somewhere along perpendicular NN'.

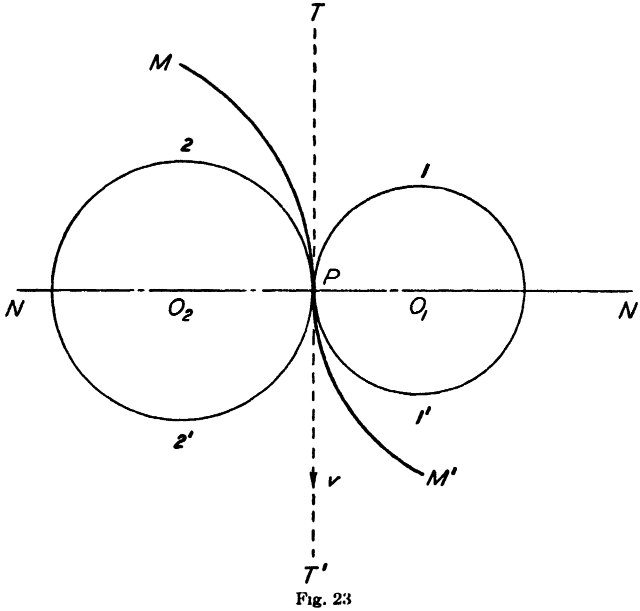


Fig. 23

From the foregoing discussion, the following general statements should be carefully noted.

(*a*) The instantaneous motion of a point is independent of its path.

(*b*) The instantaneous motion of a point is assumed to be one of rotation.

(*c*) The instantaneous motion of a point is governed by all the rules and principles of rotation.

(*d*) The instant center of a single point of a body will be somewhere along a line drawn, from the point, perpendicular to its direction of motion. It can not be definitely located on that perpendicular.

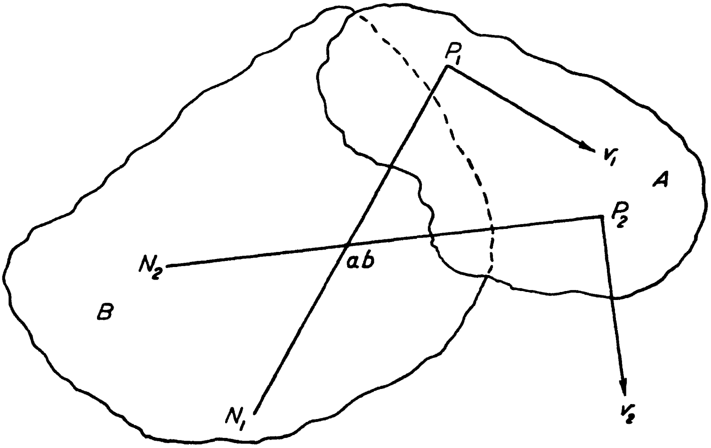


Fig. 24

(*e*) The instant center of the relative motion of one body with respect to another body may be definitely located.

To prove statement (*e*), refer to Fig. 24 in which the instantaneous motion of body A, as it moves relative to body B, is given by the motions of two of its points, P_1 and P_2. (*Note.* In general, the motion of a body having plane motion is determined by the motion of any two of its points.) The vector P_1v_1 represents the instantaneous motion of P_1, while the vector P_2v_2 represents the instantaneous motion of P_2. From statement (*d*) it is evident that the instant center of P_1 must lie somewhere on line P_1N_1, and likewise the instant center of P_2 must lie somewhere on line P_2N_2; for P_1N_1 is drawn perpendicular to P_1v_1 and P_2N_2 is drawn perpendicular to P_2v_2. But P_1 and P_2 are points of body A and hence must rotate at the instant about the same point as a center; for the center of rotation of a body is the center about which every point of that body must rotate. Therefore P_1 and P_2 must rotate about the instant center *ab*, which is the point of intersection of P_1N_1 and P_2N_2 and hence the only point which satisfies the above condition. It has been proved, then, that the point *ab* is the instant center of the relative motion of body A with respect to body B at this phase, and only at this phase.

If, at all succeeding phases of the relative motion of A with respect to B, the instant center *ab* should be found to have the same position, then A would have pure rotation with respect to B, and *ab* would be an instant center that is also a permanent or fixed center.

Other Important Characteristics of an Instant Center. When two bodies, such as the crank A and the connector B of Fig. 22, are connected by a pin joint, the turning pair thus created permits A and B to turn or rotate relative to each other. The axis of rotation is of course the axis of the cylindrical pin which is used in the pin joint. But this axis of rotation is not only the axis of the pin or of its cylindrical bearing surface; it is also the axis of the other cylindrical element of the pair. Therefore the axis is a line which belongs to both elements of the pair and hence is a common line and the only common line which these elements have. But these pairing elements belong to the two links which are connected, hence their axis of rotation is the only line which they have in common; that is, the axis of the pin joint of such links as A and B of Fig. 22 is the only line that belongs to both A and B.

As stated previously, an axis of rotation always projects to the plane of projection (which is used in a graphic solution and which, it should be remembered, is parallel to the plane of motion) as a point which is called the center of rotation or, more specifically, the instant center; for, as already explained, all centers of rotation are instant centers. Hence it is evident that an instant center (such as point *ab*) of two bodies having relative motion with respect to each other is the only common point which the two bodies have at the instant. It may be used as a point of either of the two bodies. Since any point can have but one motion at a given instant, an instant center is a point which has the same motion whether it is treated as belonging to one of its two bodies or the other. Since it has the same motion, it is a point at which the two bodies concerned have no relative motion.

Kennedy's Theorem. It was noted in the article which dealt with permanent and fixed centers, that those instant centers of Fig. 22 which are either permanent or fixed are exposed directly in the figure, but that those which are neither permanent nor fixed are not exposed in the figure. It is possible then, in either the projection diagram or the skeleton diagram of any mechanism, to locate the permanent and fixed types of instant centers from an inspection of the figure. Those instant centers which are neither permanent nor fixed may be located by the application of a theorem which was proposed by Professor Kennedy.

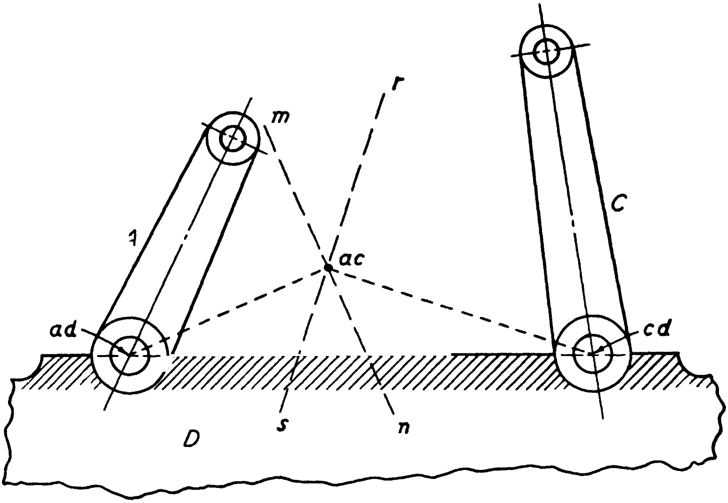


Fig 25

Kennedy's Theorem states that, in a mechanism having plane motion, the three instant centers (or centros) of the relative motion of any three links of the mechanism must fall on the same straight line.

For the proof of this theorem, Fig. 25 is introduced. This figure shows the driver *A*, the follower *C*, and the frame *D*, of a four-bar linkage. Since the theorem deals with any three links of a mechanism, the selection of these three links is sufficient for the proof at hand; the fourth link, the connector, is of no consequence.

In dealing with these three links there are evidently only three relative motions to consider: that of *A* with respect to *D*, that of *C* with respect to *D*, and that of *A* with respect to *C*. Hence there are only three instant centers involved; namely, *ad*, *cd*, and *ac*. Of these three instant centers, *ad* and *cd* are definitely located by inspection, for they are respectively the fixed centers of the relative motions of *A* and *C* with respect to the frame. The third and unknown instant center is *ac*, which is the center of the relative motion of *A* with respect to *C*. In order to have *ac* in view on the figure, let us locate it tentatively as shown. The instant center *ac* is a point which is common to links *A* and *C;* it will be considered at first as a point of link *A* and next as a point of link *C*.

All points of link *A* rotate, relative to the frame, about the instant and fixed center *ad*. Therefore, when *ac* is taken as a point of *A*, it will move in a circular path about *ad* as a center, and with the line *ad-ac* as a radius. Since the direction of motion of a point in rotation is at any instant along a tangent to its path, as a point of *A*, *ac* must move along the line *mn*, which is a perpendicular to *ad-ac* at *ac*. (A tangent to a circle is perpendicular to the radius drawn to the point of tangency.)

Let us now consider *ac* as a point of link *C*. Since all points of *C* rotate relative to *D* about the instant and fixed center *cd*, point *ac* is now rotating about *cd* with the line *cd-ac* as its radius. Hence, as a point of *C*, the direction of motion of *ac* is perpendicular to *cd-ac*, or along the line *rs*. But, like any point, *ac* can have but one direction of motion at any instant. Therefore, point *ac* must be so placed or located that lines *mn* and *rs* are coincident. This can only take place when the radii *ad-ac* and *cd-ac* are in the same straight line. Since *ad* and *cd* are fixed points, *ac* (which of course is on each of its radii) can be located at no other place than somewhere along the line *ad-cd*. Hence the three instant centers of the relative motion of any three links of a mechanism must fall on the same straight line.

Number of Instant Centers in a Mechanism. The number of instant centers involved in study of the relative motion of the links of any mechanism is given by the following formula:

$$N_c = \frac{N_L(N_L-1)}{2} \tag{18}$$

in which

N_c = the number of instant centers

N_L = the number of links in the mechanism

Thus, in a four-bar linkage in which $N_L = 4$,

$$N_c = \frac{4(4-1)}{2} = 6 \text{ instant centers}$$

Instant Center Polygon. The number of instant centers can also be obtained graphically, and the figure involved is of further use later on. Draw a polygon having as many sides as there are links in the mechanism. Draw all diagonals of the polygon. Name the vertices of the polygon by the letters representing the links. There will then be as many instant centers in the mechanism as there are lines (sides and diagonals) in the polygon. And each line of the polygon will represent a particular instant center. Thus in Fig. 26, which is drawn for the case of a four-bar linkage, there are six lines, each line representing one of the six possible instant centers of the mechanism, and diagonal *ac*, for instance, represents instant center *ac*. In locating instant centers, the lines of such a polygon as this should be made dotted at first. Then, as each instant center is located on the skeleton diagram of the mechanism, the line which represents it on the polygon is made solid. In this way, the unknown instant centers are evident at a glance.

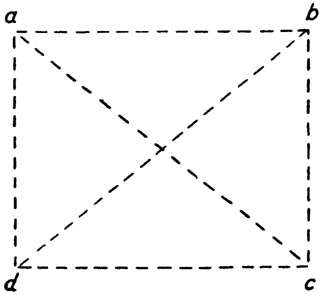


Fig 26

It should also be noted that such a polygon as Fig. 26 is always composed of a series of triangles. Each triangle has the names of three links at its vertices, and its three sides represent the instant centers of the relative motion of these three links. Hence, from the statement of Kennedy's theorem, the three instant centers given by the sides of each triangle must fall on the same straight line. This idea will be made use of in the work which follows.

Location of Centros in a Four-Bar Linkage. The method of procedure that is followed in the location of the instant centers of this four-bar linkage is perfectly general and is applicable to all linkwork.

The mechanism under consideration here is a four-bar linkage like that of Fig. 22. It will be noticed from Fig. 27(*a*) that a skeleton diagram or layout of the mechanism is first drawn and the links named. Here the lines of the links are marked *A*, *B*, *C*, and *D*, representing the driver, the connector, the follower, and the frame, respectively. Having completed this layout, the instant center polygon, Fig. 27(*b*) is drawn.

Now those permanent and fixed instant centers which can be located directly from an inspection of the layout are picked out. It should be remembered that the center of every pin joint is an instant center, and also that an instant center is a point that is common to two links at the instant. The pin joint by which *A* is attached to the frame is the instant center of the relative motion of *A* with respect to *D;* it is marked *ad,* as shown in Fig. 27(*a*), and is immediately registered on the instant center polygon, Fig. 27(*b*), by making a solid or full line of that line of the polygon which runs from *a* to *d.* In a similar manner, the instant centers at the other pin joints, namely *ab, bc,* and *cd,* are lettered on the layout, having been determined by inspection, and the corresponding lines on the polygon are made solid. At this stage in the solution of the problem, the polygon will appear as shown in Fig. 27(*c*), and the four lines shown solid in the polygon tell at a glance the instant centers that have been located on the layout.

An inspection of Fig. 27(*c*) shows that two more, and only two more, instant centers are still to be obtained. These are shown by the dotted lines to be *ac* and *bd.* Now the triangle *abc* of the polygon of Fig. 27(*c*) has three sides which represent the three instant centers of the relative motion of links *A*, *B*, and *C*. These three instant centers are *ab, bc,* and *ac,* and Kennedy's theorem instructs us that they must fall on the same straight line. Since a straight line is determined by two points, the known points *ab* and *bc* will determine the line on the skeleton diagram upon which the unknown point *ac* will fall. Hence *ac* will lie on the line *ab-bc* of the skeleton layout. Likewise *ac* is the only side of triangle *adc* of Fig. 27(*c*) that is dotted. Hence *ac* will lie on the line of Fig. 27(*a*) determined by the two known instant centers *ad* and *cd*. This is the line *ad-cd*. This gives a second line upon which the point *ac* will fall. Since a point which is located on two lines must be at the intersection of those two lines, *ac* is at the intersection of *ab-bc* and *ad-cd* as shown in Fig. 27(*a*).

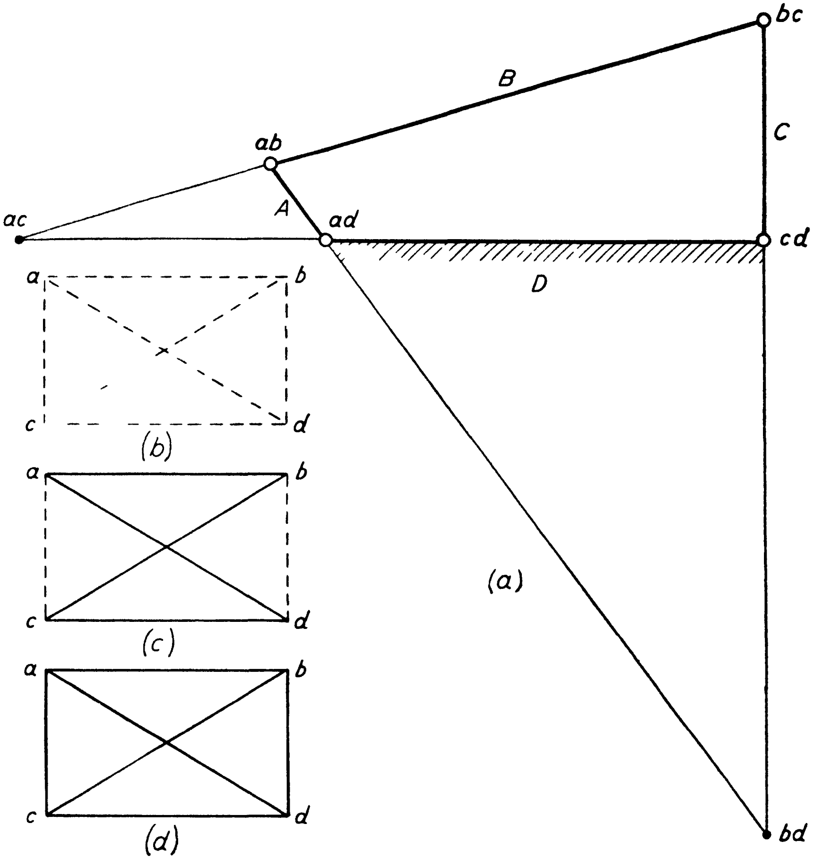


Fig 27

This leaves but one instant center, *bd*, that is still to be located on the layout. It will be noted that *bd*, a line of Fig. 27(*c*), is the third and unknown side of two triangles, *abd*, and *cbd*. From triangle *abd*, instant center *bd* must lie on a line of the layout which is determined by the instant centers *ab* and *ad*. This is line *ab-ad* of the skeleton diagram. Also, from triangle *cbd*, instant center *bd* must lie on a line of the layout which is determined by *bc* and *cd*. This is line *bc-cd* of the skeleton diagram. Since *bd* lies on both lines as given, it can only be at one point and that is at their intersection. The lines are of course extended (whenever necessary) until they meet at a point, and this point is given the name, *bd*. Since *ac* and *bd* have now been located, the lines on the polygon are made solid and Fig. 27(*c*) now appears as Fig. 27(*d*), indicating that all the possible instant centers of the mechanism have been found and are located on the skeleton diagram.

Of course this filling in of the lines on the instant center polygon is all done on one polygon by the student, so that the polygon of Fig. 27(*b*) eventually is composed of nothing but solid lines like Fig. 27(*d*). Three polygons are used in the text to more clearly show the successive steps. The importance of the instant center polygon is more evident in mechanisms having a greater number of links than the four-bar linkage.

Note. Fig. 27(*a*) indicates that, in our layouts, small open circles are used for instant centers which are located at pin joints, and small solid circles are used for all other instant centers. The small solid circle will also be used in the future to represent a given point on a link, but when so used it will be named by a single letter.

The student should make a drawing of the linkage of Fig. 27(*a*) together with the polygon of Fig. 27(*b*) and locate the instant centers without the aid of the text. He should follow the same procedure with all succeeding graphical solutions given in the text.

Directional Relation in a Four-Bar Linkage. It often occurs in a four-bar linkage that there are certain phases of the cycle of operation of the mechanism at which the connector passes between the fixed centers of the driver and follower. Such is the case with the linkage of Fig. 27 when it reaches the phase established in Fig. 28. The latter is generally referred to as a *Crossed-Link* phase, in contrast to the *Open-Link* phase shown in Fig. 27. In the latter it will be seen that the line of the connector *B* (extended if necessary) does not pass between the fixed centers of driver or follower; on the other hand it cuts, or passes through, the line of fixed centers, *ad-cd*, in such a manner that the latter is divided externally, rather than internally, into segments *ac-ad* and *ac-cd*.

The above facts determine the *Directional Relation* of driver and follower in the four-bar linkage. Thus in the open-link phase of Fig. 27 it will be noted that if the driver *A* turns or rotates in a clockwise direction (the direction in which the hands of a clock rotate) the follower *B* rotates in the same direction. This is always true if the connector does not pass between the fixed centers of driver and follower. On the other hand, in the crossed-link phase of Fig. 28 it will be noted that, with the driver *A* still assumed as rotating clockwise, the follower will be compelled to rotate counterclockwise (opposite in direction to that in which the hands of a clock rotate). This statement holds true in all phases when the connector passes between the fixed centers of driver and follower. Should such a condition never occur, the driver and the follower will always turn in the same direction.

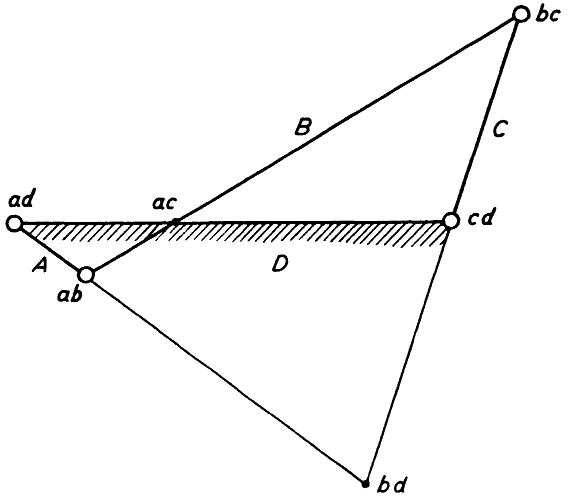


Fig 28

Since Fig. 28 deals with the same linkwork as Fig. 27, it is evident that the same instant-center polygon can be used in each case; hence it is not repeated in Fig. 28. The method of procedure in locating the instant centers of the phase of Fig. 28 is identical to that used for the open-link phase of Fig. 27. Since the same letters are assigned to the individual links, the student need only reread the preceding article, applying it to Fig. 28 instead of Fig. 27. When all of the instant centers in this or any other diagram have been found, check them carefully to see that the three instant centers of the relative motion of any three links (those given graphically by the sides of any triangle of the polygon) fall on the same straight line. The straight line that they fall upon may not be one that has been used to locate them; in fact such a line serves as a better check on the accuracy of the work.

Location of an Instant Center of a Sliding Block. A mechanism similar to the linkwork of Figs. 27 and 28 is shown in Fig. 29. The relative lengths of the links are such that while crank *A* rotates in one direction continuously, follower *C* oscillates about its fixed center so that its pin at the outer end moves back and forth along the circular path between points *m* and *n*. Driver *A* is at the dotted position shown by A_1 when the pin is at *n*, and it is at A_2 when the pin is at *m*; the driver and connector are coincident in each case. The instant center of the relative motion of all points of link *C* with respect to the fixed link, or frame *D*, is, as previously seen, the fixed center *cd* that is shown on the figure.

Now let us suppose that follower *C*, as it appears in Fig. 29, is dispensed with, and in its place there is substituted the sliding block *C* of Fig. 30, which might be thought of as merely the outer end of the original follower. The frame is extended and provided with a slot into which slider *C* fits. The mean radius of the slot is taken equal to the length of the follower of Fig. 29. By means of this slot the frame constrains the motion of the slider so that the slider has the same motion in the mechanism of Fig. 30 that the end of the original follower has in Fig. 29. It is evident that *cd*, the instant center of the relative motion of the slider with respect to the frame, is at the same position in Fig. 30 as in Fig. 29.

Let us imagine that the radius of the slot in the frame of Fig. 30 is gradually increased. This would cause the instant center *cd* to recede farther and farther and the arc to become flatter and flatter until finally the instant center would be at an infinite distance from the arc, and the latter would be a straight line. Such a condition produces the slider-crank mechanism of Fig. 31, in which the arc *mn* of Figs. 29 and 30 has become the straight line *mn*. Since the direction of the slider is now along the straight line *mn*, the instantaneous radius of the slider (at the phase as given in the figure) may be drawn through the center of its pin joint perpendicular to line *mn*. The instant center of the relative motion of the slider with respect to the frame (or of the frame with respect to the slider) will be at an infinite distance in

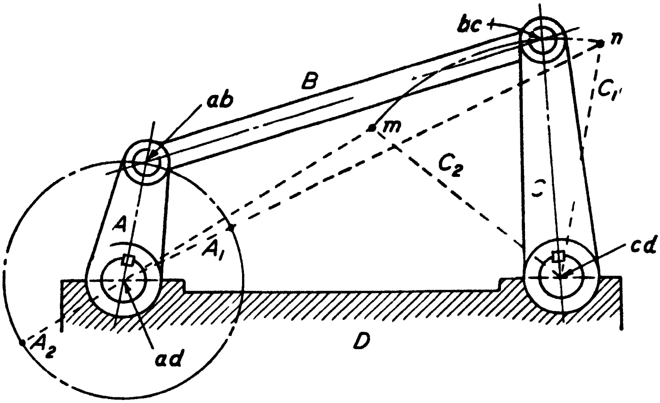


Fig 29

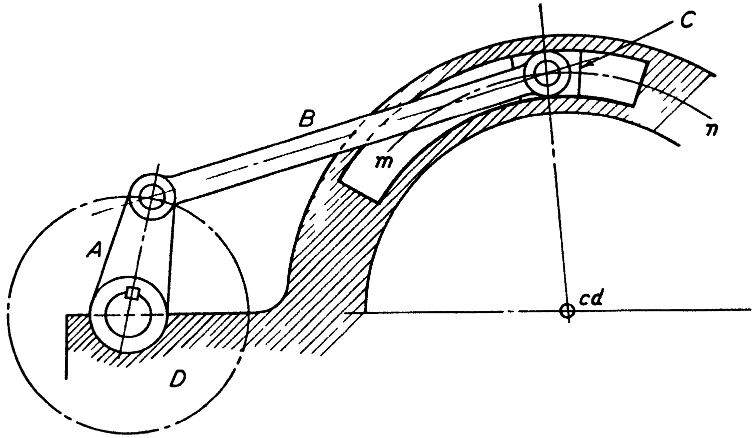


Fig. 30

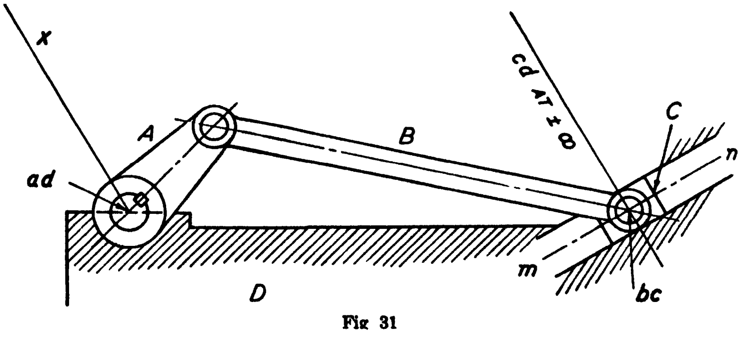


Fig 31

either direction from point *bc*, out on this perpendicular to *mn*. This fact is indicated on the figure by stating that *cd* is at ± ∞ (plus or minus infinity).

When the frame of the mechanism constrains the slider so that its path is along a horizontal line drawn through the fixed center *ad* of the crank, the slider-crank mechanism of Fig. 32 is produced. This is the familiar reciprocating steam engine mechanism. It should be noted that the instantaneous radius of the slider is here drawn through point *bc* perpendicular to the horizontal. It therefore becomes a vertical line and, of course, out on it at an infinite distance either above or below *mn*, the instant center *cd* is located.

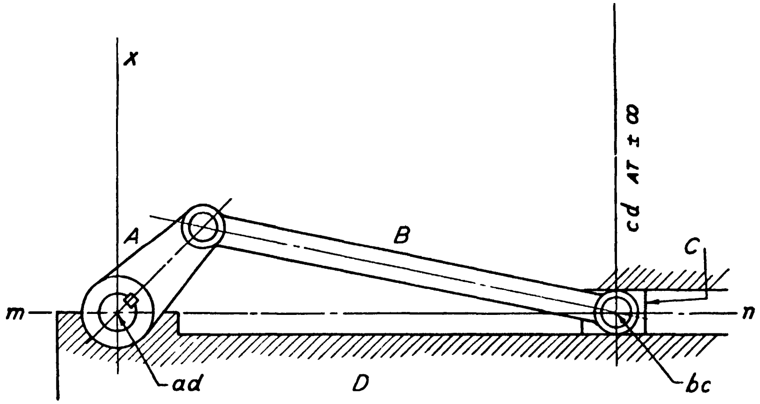


Fig. 32

Suppose that, in the case of either Fig. 31 or Fig. 32, a line must be drawn from *ad*, the instant center of the crank, to *cd*, the instant center of the slider. In such a case, a line is drawn through *ad* parallel to the instant radius which goes out to *cd* at ± ∞. The reason for doing this is that two parallel lines intersect at infinity. The student should carefully note on both figures the line drawn from *ad* to point *cd* at infinity. These lines may be identified in Figs. 31 and 32 by the letter x placed upon them.

Location of Centros of the Slider-Crank Mechanism. The slider-crank mechanism will familiarize the student with the manner in which to deal with an instant center which is located at infinity (∞), in that it will demonstrate how such a center can be used to help determine lines upon which other instant centers will fall.

The skeleton diagram of the mechanism is first drawn as shown in Fig. 33(*a*), and the links are then named by their letters. Here *A* is the crank; *B*, the connecting rod; *C*, the slider; and *D*, the frame. The direction of motion of the slider is noted in particular and is seen to be along the so-called center line of the engine, which is a horizontal line in this case. The slider (crosshead or piston) moves along this line because it is constrained by the frame to do so.

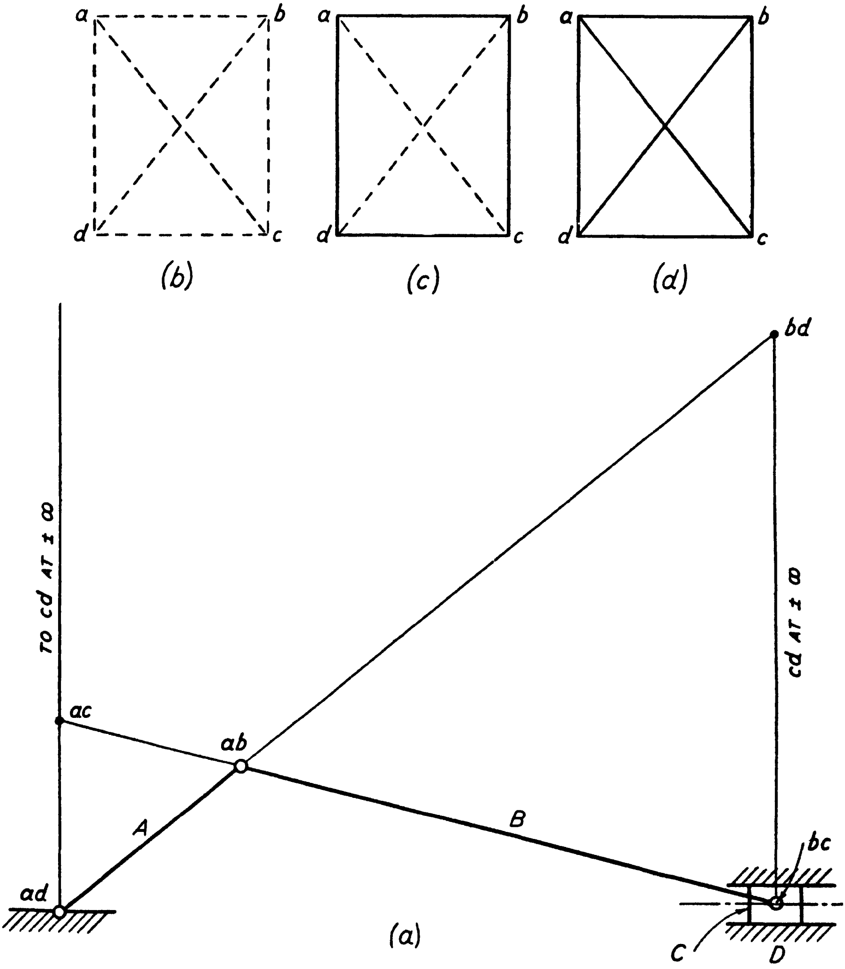


Fig. 33

Since the mechanism contains four links, an instant-center polygon of four sides, Fig. 33(*b*), is now drawn with all of its diagonals and sides given by dotted lines. The vertices of the polygon are named by the letters (lower case) representing the various links of the mechanism so that, as has been stated previously, each line of the instant-center polygon represents an instant center. Thus the top line, *ab* of Fig. 33(*b*), represents the instant center of the relative motion of link *A* with respect to link *B*. As the work proceeds, the dotted lines of Fig. 33(*b*) will be made solid as the corresponding instant centers are located on Fig. 33(*a*). As before, the instant-center polygons of Fig. 33(*c*) and 33(*d*) are given only for instructional purposes to show the progress in the change of Fig. 33(*b*) as the instant centers become known.

The next step is to locate all of those instant centers which can be determined from an inspection of the layout. In this manner, the pin joints, which are used as connections between *A* and *D*, *A* and *B*, and *B* and *C*, afford the instant centers *ad*, *ab*, and *bc* respectively. These are named on the layout and are represented by small open circles because they are permanent centers; the lines which represent them on the instant-center polygon are made solid.

One more instant center can now be found by inspection and that is the instant center *cd* of the relative motion of the slider *C* with respect to the frame *D*. Since this relative motion of the slider is one of translation along a horizontal line, as given, a line is drawn through *bc* perpendicular to this horizontal line and becomes, in this case, the vertical line upon which the instant center *cd* is indicated as located at $\pm\infty$; for instantaneous motion is, and may always be assumed as, rotation; and therefore translation is, at the instant, rotation about a center at infinity. The center *cd* is said now to be located and so is made solid on the polygon. At this stage, the latter is as shown in Fig. 33(*c*), which indicates by the dotted lines *ac* and *bd* that the two instant centers *ac* and *bd* are yet to be found.

Fig. 33(*c*) not only shows what instant centers must still be obtained; it also points the way by which they may be obtained. For instance, this figure is examined to find a dotted line that is common to two triangles and that is the only dotted side either triangle contains. The side *bd* is such a line. Hence the instant center *bd* in the layout of Fig. 33(*a*) will not only lie on the line determined by the known centers *ab* and *ad* (triangle *abd*) but also on the line determined by the known centers *bc* and *cd* (triangle *bcd*). Therefore *bd* must be at the intersection of lines *ab-ad* and *bc-cd* of Fig. 33(*a*). Extending these lines until they meet, *bd* is located as shown.

In a similar manner, from Fig. 33(*c*) it is seen that instant center *ac* lies simultaneously on two lines, one of which is determined by points *ab* and *bc* (triangle *abc*) while the other is determined by points *ad* and *cd* (triangle *acd*). The former of these two lines is the line of the link B. The latter of these two lines must be drawn on the layout. Since it is determined by points *ad* and *cd*, *ad* must be connected to a point, *cd*, located at infinity on line *bc-bd*. Therefore this line *ad-cd* is drawn through *ad* parallel to *bc-bd*. Extending *ab-bc* until it intersects *ad-cd*, the point *ac* is located at their intersection. Since centers *bd* and *ac* are now shown on the layout, their lines in the polygon are made solid and Fig. 33(*c*) becomes as shown in Fig. 33(*d*).

Location of Centros in a Six-Link Mechanism. A skeleton diagram of an oscillating-arm quick-return mechanism, together with all its instant centers, or centros, is illustrated in Fig. 34. The driver of the mechanism is the crank A, which rotates about the fixed center *af*. A sliding block is pinned to the end of the crank at the permanent center *ab*. This block slides in a slot in the swinging or oscillating arm C. In this manner, the rotation of A causes the link C to oscillate about its fixed center *cf*. The oscillating arm drives the slider E through the connector D. The slider E is constrained by the fixed guides of the frame to move back and forth in a straight line, as shown in the figure.

Since this mechanism is made up of six links, the instant-center polygon will be a hexagon, as shown in Fig. 35. It is advisable, although not absolutely necessary, to use a regular hexagon. Having drawn in all the diagonals, the polygon will be made up of fifteen lines. Therefore a six-link mechanism will have fifteen instant centers at any phase of its cycle that may be investigated. To verify this number of instant centers, formula (18) can be applied. Thus with $N_L=6$,

$$N_c=\frac{N_L(N_L-1)}{2}$$

$$=\frac{6(6-1)}{2}=\frac{6\times 5}{2}=15 \text{ instant centers.}$$

The method by which the instant centers of this mechanism are located is somewhat similar to that used with the preceding linkages. The solution of this problem of course is longer and somewhat more complicated due to the greater number of centros involved.

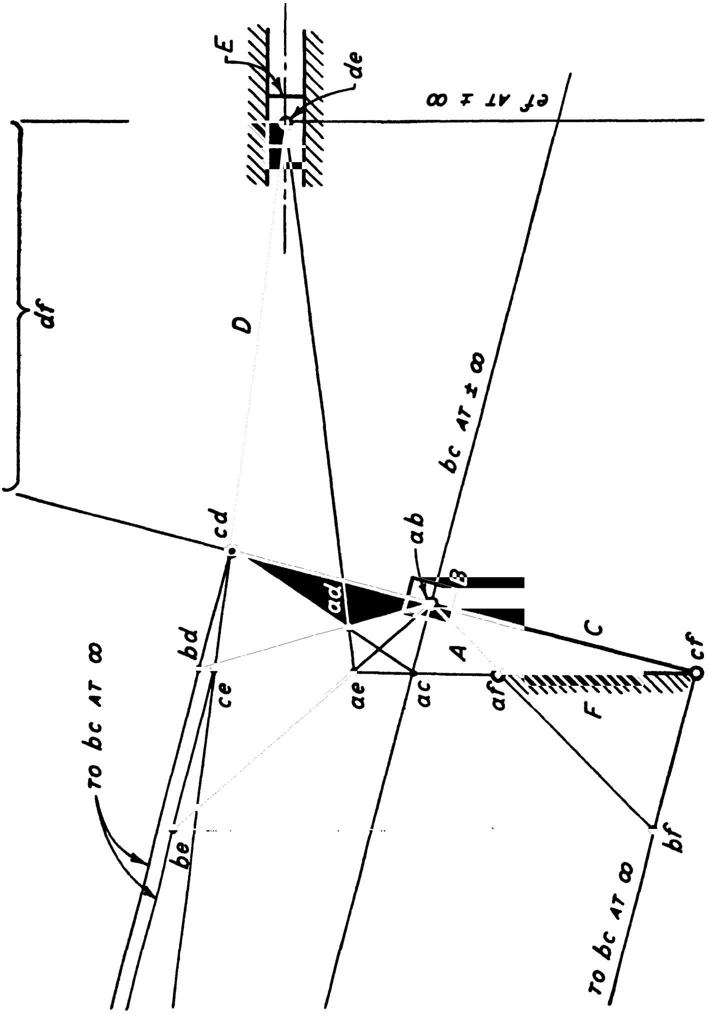


Fig. 34

Having drawn the skeleton diagram and named the links thereof, and then having drawn the instant-center polygon, lettering the vertices, the next step (as always) is to locate all of the instant centers that can be determined by inspection. These are seen to be the two fixed centers *af* and *cf*, the three permanent centers *ab*, *cd*, and *de*, and the two centers at infinity, *bc* and *ef*. These two centers at infinity are located (as in previous explanations) on lines drawn through the centers of the pin joints of the sliders and perpendicular to the direction in which the sliders are moving at the given instant which is represented by the phase of the mechanism as given in the layout. These seven instant centers which have been obtained by inspection of the layout are now lettered on the layout and the lines that represent them on the polygon are made solid. Numbers from *1* to *7* inclusive are assigned at random to these instant centers and placed along the lines of the polygon. (It is thought that this assignment of numbers will be helpful to the student from an instructional standpoint. In the solution of like problems, the student in all probability will not find such an assignment of numbers to be necessary.)

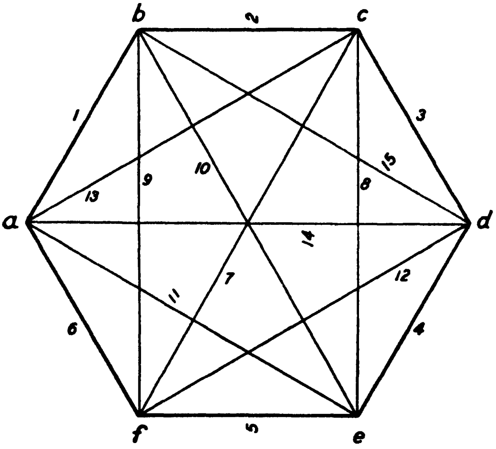


Fig 35

The remainder of the instant centers, numbers *8* to *15* inclusive, are now to be located. These of course are located by the aid of Kennedy's Theorem as applied by the instant-center polygon, Fig. 35. It will be remembered that in Fig. 35 every triangle represents three instant centers that must fall on the same straight line of the layout; so if two of them are known (solid lines), a straight line on the layout is determined by these two, and the third (dotted line), which is unknown, must then fall on that line. Referring to Fig. 35 as developed so far, it will be seen that the next instant center which can be obtained is number 8, or *ce*. It is shown by the triangles *cde* and *cef* of the polygon to be on the lines *cd-de* and *cf-ef*. These two lines are drawn or extended on the skeleton diagram so as to give *ce* at their intersection.

A sequence of steps by which all these instant centers from *8* to *15* inclusive may be located is given in the following tabulation. In using this table, read the heading of the first column, then the material on the first line below it; the next column head, then the material on the first line below that; the third column head, and so on until the entire first line is read. The operation is repeated for each succeeding line. For example, the material of the last row reads as follows: "Instant center No. 15, which is *bd*, is shown by the triangles *abd* and *bcd*, to be at the intersection of the lines *ab-ad* and *bc-cd*."

Sequence of Steps

Instant center No.	which is	is shown by the triangles	to be at the intersection of the lines
8	*ce*	*cde* and *cef*	*cd-de* and *cf-ef*
9	*bf*	*abf* and *bcf*	*ab-af* and *bc-cf*
10	*be*	*bce* and *bef*	*bc-ce* and *bf-ef*
11	*ae*	*abe* and *aef*	*ab-be* and *af-ef*
12	*df*	*def* and *cdf*	*de-ef* and *cd-cf*
13	*ac*	*abc* and *acf*	*ab-bc* and *af-cf*
14	*ad*	*acd* and *ade*	*ac-cd* and *ae-de*
15	*bd*	*abd* and *bcd*	*ab-ad* and *bc-cd*

Note. Having studied carefully the material of this article in conjunction with Figs. 34 and 35, the student should make a duplicate drawing of the linkage of Fig. 34 and locate the instant centers thereon with the aid of the text. Following this a solution should be completed without use of the text.

Centrodes. The four-bar linkage of Fig. 36 is one in which the relative lengths of the various links permit the driver *A* to produce complete rotation of the follower *C*. Since *ad* and *cd* are fixed centers, it is evident without drawing in a series of successive positions of the linkage that the path of the instant center *ab* is a circle of radius *ad-ab* and that the path of the instant center *bc* is a circle whose radius is *cd-bc*. These circles are shown on the figure. Such a path which is followed by an instant center is called a *Centrode*. It is a line traced on the stationary member of the mechanism by an instant center as the latter takes its successive positions throughout a cycle. It is retraced by the instant center during each succeeding cycle. The centrode of a fixed center is evidently a point on the frame. The centrode of point *bc* of Fig. 33(*a*) is a straight line of a length equal to twice the length or throw of the crank. (The length of a crank is taken as the distance from the center of the crank shaft to the center of the crank pin.) The centrode of *bc* of Fig. 29 is the arc *mn*.

A centrode may be a complex curve as is the case with the centrode of *bd* of Fig. 36 which is also shown on the figure. In order to obtain the successive positions of *bd* that will define its centrode, successive phases of the mechanism are drawn. This is accomplished by taking different positions of the driver *A*, and locating the other links in their corresponding positions. The position of the instant center *bd* is then found for each phase taken. Since the known instant centers *ab* and *ad* involve the three links *A*, *B*, and *D*, whose only other instant center is *bd*, according to Kennedy's theorem, *bd* must fall on the line determined by *ab* and *ad*. This line is always a position of the driver *A*. For the same reason, *bd* must always fall on the line determined by the instant centers *bc* and *cd*, which line is a position of the follower *C*. Hence *bd*, at any phase or position of the mechanism,

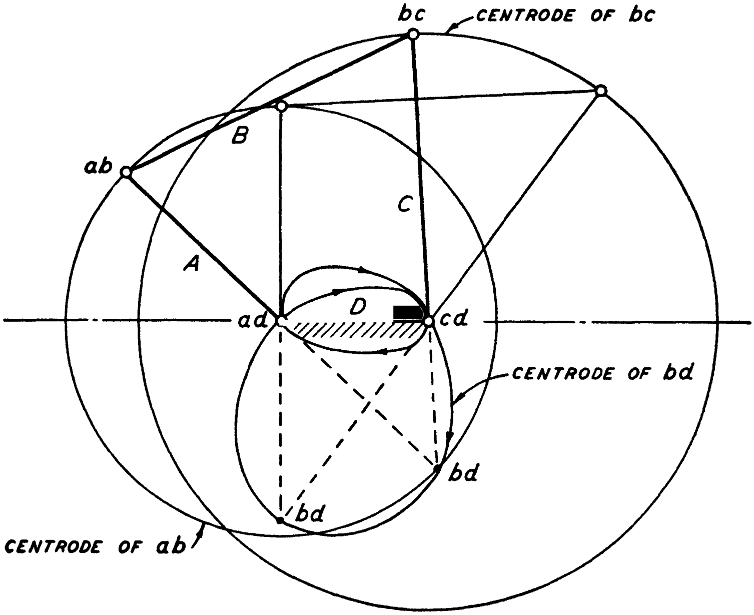


Fig 36 Centrodes

is at the intersection of the driver *A* and the follower *C*. Two phases of the mechanism are shown in Fig. 36, and *bd* is located for each phase. Having located a sufficient number of such points to determine the path, a line connecting them in the proper order is the centrode of *bd* that is traced by it on the frame.

Since a centrode is the path which an instant center traces on the stationary link or frame, that path depends upon which link of the mechanism is taken as the stationary link. It is a fact that any one of the links of a mechanism can be the stationary link, and the relative motions of the various links of the mechanism with respect to each other are the same no matter what link is assumed to be fixed.*

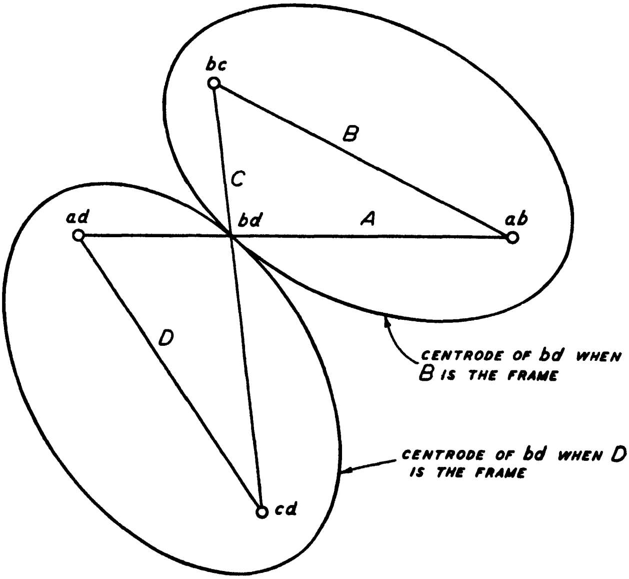

bc
B
C
ad
bd
A
ab
D
CENTRODE OF bd WHEN B IS THE FRAME
CENTRODE OF bd WHEN D IS THE FRAME
cd

Fig. 37. Centrodes

In Fig. 37 is shown a special case of a four-bar linkage in which the link *A* is equal in length to link *C*, and link *B* is equal in length to link *D*. When *D* is taken as the stationary link, the centrode of *bd* is the lower ellipse; while with *B* as the stationary link, the centrode of *bd* is the upper ellipse. Due to the equal lengths of the links as noted, this mechanism is exactly the same whether *B* or *D* is considered as

---

*From a kinematic standpoint then, the mechanism remains the same, but from a practical standpoint its form and usage are generally quite different. The change in a mechanism, whereby one link is substituted for another as the fixed link, is called an *Inversion* of the mechanism.

the frame. Hence the two elliptical centrodes of *bd* are equal ellipses. In most cases, such a pair of centrodes would be unlike curves, but in all cases they are tangent to each other at a common position of the instant center.

Axode. The plane of projection which is used in all of the layouts or skeleton diagrams, as stated previously, is taken parallel to the plane of motion. This insists upon an instantaneous axis projecting to the plane of the layout as a point, which is an instant center. Hence every point on a centrode is the projection of a position of an instant axis, which must at all times lie in a surface which projects to the plane of the layout as a centrode. Such a surface is called an *Axode.* An axode may also be defined as the surface generated by an instant axis as the latter takes its successive positions during a cycle. The axode of the instant axis which projects to the plane of Fig. 36 in the center, *ab,* is evidently a right circular cylinder, as is also the axode of *bc*. In Fig. 37 the axodes are elliptical cylinders, and in Fig. 33(*a*) the axode of the instant axis *bc* is a plane surface.

QUESTIONS AND PROBLEMS

1. How many instant centers will there be in a mechanism which is composed of five links? *Ans.* 10

2. How many instant centers will there be in a mechanism which is composed of three links? *Ans.* 3

3. Classify instant centers.

4. What is the name of that point which is the only point in common between two links of a mechanism at an instant?

5. At what point have two links of a mechanism no relative motion?

6. Two links of a mechanism are named by the letters *D* and *E*. How does the relative motion of *D* with respect to *E* differ from the relative motion of *E* with respect to *D*?

7. State Kennedy's theorem.

8. In the slider-crank mechanism of Fig. 38, *A* is the frame; *B*, the crank; *C*, the connecting rod; and *D* is the slider. Reproduce this layout with the links named as given and with a crank length of 1 inch and a connecting-rod length of 4 inches. Locate on the layout all of the instant centers of this mechanism. Place no dimensions on the layout drawing.

9. In the drawing for the preceding problem, show the centrodes of *bc* and *cd*.

10. Reproduce the four-bar linkage of Fig. 39 with the links named as shown. Let the driver, *A*, be 1 inch in length and the follower, *B*, 1¾ inches in length. Locate upon the layout all the instant centers of this mechanism. Always use the instant-center polygon.

11. What is an axode?

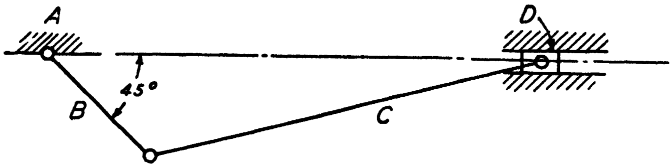


Fig 38

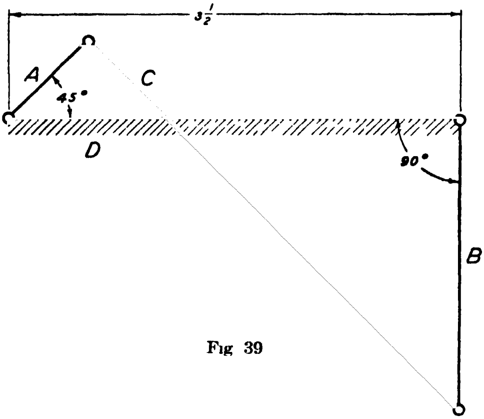


Fig 39

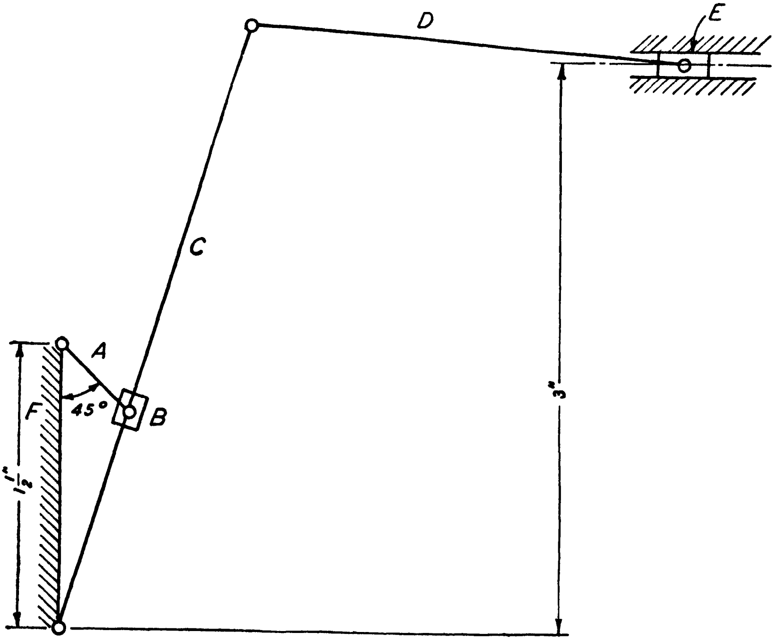


Fig. 40

12. Required: to locate all the instant centers of the oscillating-arm quick-return motion of Fig. 40. Let the lengths of the lines of the links *A*, *C*, and *D* be as follows: *A*, ¾ inch; *C*, 3¾ inches; *D*, 3⅜ inches.

13. Show the centrodes of the instant centers *ab* and *cd* on the layout of Problem 12.

14. Name the instant centers of Fig. 40 that are
(*a*) permanent but not fixed
(*b*) permanent and fixed
(*c*) neither permanent nor fixed.

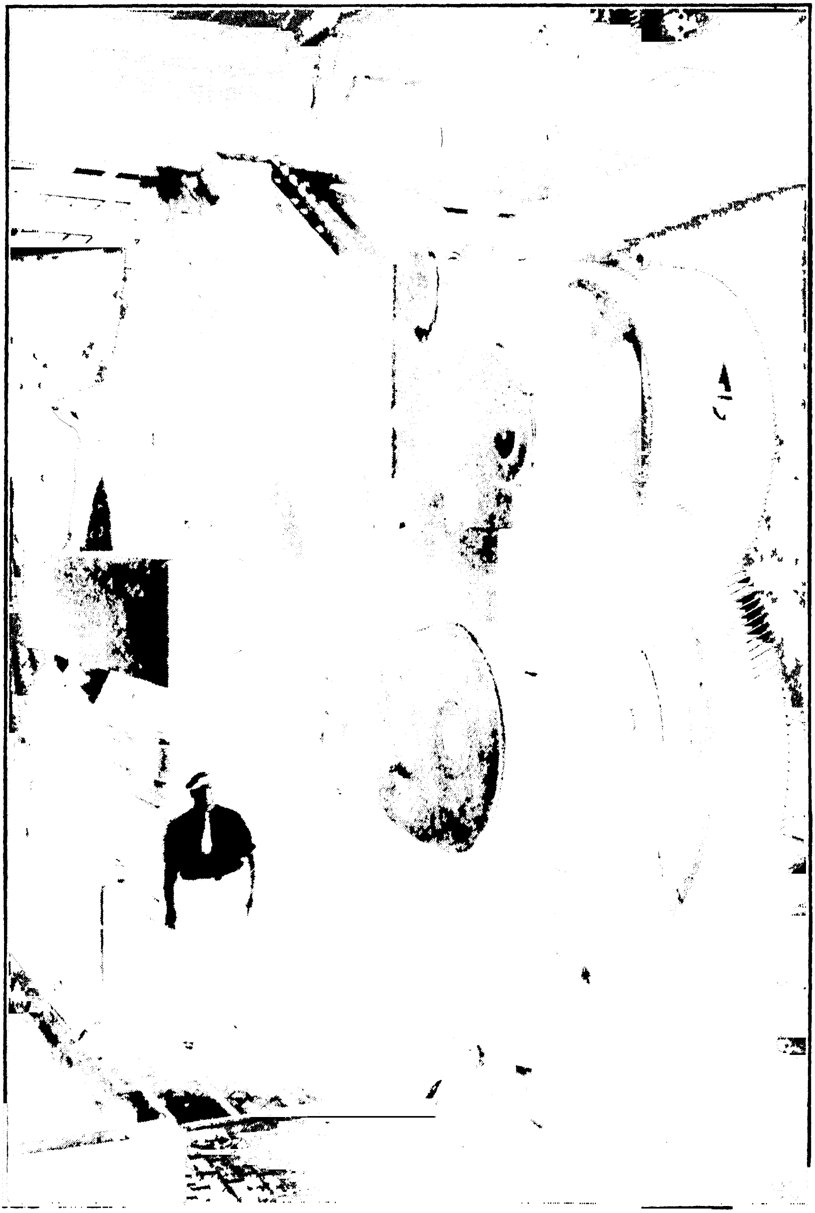
15. With the dimensions of the links as given in Problem 12, make a layout of the mechanism of Fig. 40 for that phase in which the driver *A* is in a horizontal position and extends outward to the left of its fixed center.

16. Name the instant centers of Fig. 56 that are
(*a*) permanent but not fixed
(*b*) permanent and fixed
(*c*) neither permanent nor fixed.

17. Draw the linkage of Fig. 56 at the phase in which the driver *A* is at an angle of 45 degrees with the horizontal as shown. Locate the permanent center *af* as follows: 1⅝ inches below *cf*, 15/16 inch to the left of *cf*, and 11/16 inch to the right of *de*. Locate *bc* at a distance of 1 inch from *cf*. The links have the following lengths: *A* = 5/16 inch, *B* = 1⅝ inches, *C* = 1 11/16 inches, *D* = ⅞ inch. Locate thereon all of the instant centers of the linkage at this phase. Use scale, twice size.

18. Show the linkage of Fig. 56 in the position its links occupy after the driver *A* has been rotated 45° clockwise. Locate thereon all the instant centers of the linkage at this phase. Scale, twice size.

19. Same as Problem 18 except driver A has been rotated 270° clockwise.



GIANT MOTION FOR AUTOMOTIVE FRAME ASSEMBLY
Courtesy of The Falk Corporation, Milwaukee, Wis.

CHAPTER IV

RELATIVE VELOCITIES IN LINKWORK

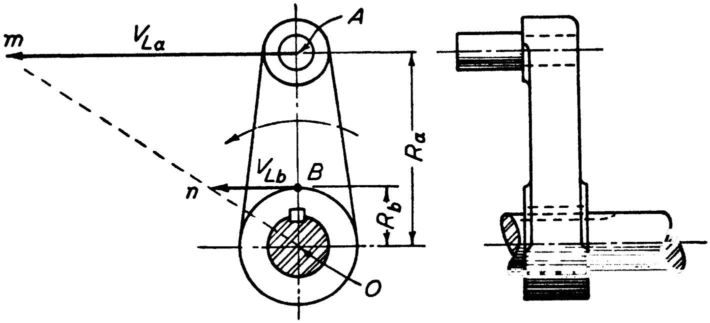
Introduction. The term "linkage" (or linkwork), as already seen, refers to that type of mechanism which employs a rigid link connector between its driver and follower. Such a mechanism is involved in one of the three fundamental modes of transmission of motion as given in the last article of Chapter II. The study of the relative velocities of this type of mechanism will be directed principally to those elementary linkages which consist of four links; for the same fundamental principles underlie the more complex linkages (the so-called compound linkages). This is due to the fact that a compound linkage is merely a group of elementary combinations which may be detected if the latter are understood.

Similar Triangles, Their Use in Determining Velocities. It often happens that the velocity of some point of a rotating body is obtained by using the known velocity of some other point of that body. In order to do this, the instant center of the link must be definitely located so that the radius of each of its two points will be known. If the link has a motion of pure rotation, its instant center will be a fixed center and hence given; otherwise, the instant center may be located by the methods previously given. Since the motion of all points is taken relative to the frame, the required instant center in all cases will be that which the link possesses with respect to the frame; and the radii, which must be used in solving for the unknown velocity, are of course the distances from the given points to the involved center.

Two methods of solution present themselves; one is mathematical and the other is graphical. In the mathematical solution the numerical values of the known velocity and the two radii are obtained and substituted in formula (**12**), as was demonstrated in the examples of Chapter II. This method is used mostly with individual links having pure rotation. The graphical solution is generally resorted to in studying the relative velocities of the various points of the different links of a linkage. The first demonstration of the use of the graphical method will deal with a single link, so that the student may note its simplicity and become thoroughly acquainted with it before its use in a problem involving a linkage.

The graphical solution makes use of the fact that formula (12) states that the linear velocities of any two points of a rotating body are directly proportional to their radii. Such a direct proportion can then be solved graphically by using either similar right triangles or similar oblique triangles; for in similar triangles the corresponding (or homologous) sides are proportional.

The application of this method is shown in Fig. 41 in which point O is the fixed center of a crank; point A is the crank pin whose radius, OA, is R_a; and point B is a point on the hub whose radius, OB, is R_b.



DRAWING SCALE 1" = 12"
VELOCITY SCALE 1" = 800 f p m.

Fig 41 Crank

Let us suppose that the known velocity of A, V_{La}, is 1200 feet per minute and that the velocity of B is required. A velocity scale of 1 inch = 800 feet per minute having been selected, the vector Am, representing the velocity of A, will be $\frac{1200}{800}$ or 1½ inches in length. Since the direction of motion is always tangent to the path of the point, this vector is laid off perpendicular to radius OA, as shown. Line Om is next drawn from the center O to the end of vector Am, and a line is now drawn through B perpendicular to OB (and OA) and terminating in Om at n. In this manner, two similar right triangles, OAm and OBn, are formed, in which Bn is the vector which represents the velocity of B to the same scale as that used for the velocity of A. This is proved as follows:

In right triangles *OBn* and *OAm*,

$$\frac{Bn}{Am}=\frac{OB}{OA}=\frac{R_b}{R_a}$$

from which, by multiplying both numerator and denominator of the first member by the velocity scale,

$$\frac{Bn\times(\textit{velocity scale})}{Am\times(\textit{velocity scale})}=\frac{R_b}{R_a}$$

But $$Am\times(\text{velocity scale})=V_{La}$$

therefore

$$\frac{Bn\times(\textit{velocity scale})}{V_{La}}=\frac{R_b}{R_a}$$

Multiplying both members of the equation by V_{La},

$$Bn\times(\textit{velocity scale})=\frac{R_b}{R_a}\times V_{La} \qquad \text{(a)}$$

From formula (12), letting V_{Lb} represent the velocity of *B*,

$$\frac{V_{Lb}}{V_{La}}=\frac{R_b}{R_a}$$

or $$V_{Lb}=\frac{R_b}{R_a}\times V_{La} \qquad \text{(b)}$$

Since, from steps (*a*) and (*b*), $Bn\times(\textit{velocity scale})$ and V_{Lb} are both equal to the same thing, they must be equal to each other. Therefore,

$$V_{Lb},\ \text{linear velocity of } B, = Bn\times(\textit{velocity scale}) \qquad \text{(c)}$$

and hence *Bn*, as a vector, represents the velocity of *B* to the same velocity scale as that by which *Am* represents the velocity of *A*.

The vector *Bn*, upon being measured, is found to be 15/32 inch in length. The magnitude of the velocity V_{Lb}, which it represents, can now be determined by substituting, in step (*c*), the values 15/32 inch and 800 feet per minute per inch, for *Bn* and the velocity scale respectively, so that

$$V_{Lb}=\tfrac{15}{32}\times 800=375 \text{ f.p.m.} \quad \textit{Ans.}$$

Now let us solve this same example by the use of formula (12). In order to do so it will be necessary that either the radii of hub, R_b, and of the crank-pin circle, R_a, or the ratio of these radii be known. The radius R_b is 3¾ inches, while the radius R_a is 12 inches. Substituting these values along with that of V_{La}, which is 1200 feet per minute, in formula (12),

$$\frac{V_{Lb}}{V_{La}}=\frac{R_b}{R_a}$$

$$\frac{V_{Lb}}{1200}=\frac{3\frac{3}{4}}{12}$$

from which
$$V_{Lb}=\frac{3\frac{3}{4}}{12}\times 1200=\frac{\frac{15}{4}}{12}\times 1200$$
$$=\frac{15}{48}\times 1200=375 \text{ f.p.m.} \quad \textit{Ans.}$$

This answer is seen to be an exact check on that obtained by the graphical method. Although this degree of accuracy is not generally attained by the graphical method, the slight error that as a rule accompanies its use is of no consequence from a practical standpoint.

Two methods of solving a velocity problem similar to that of Fig. 41 are illustrated in Fig. 42. Here a disk (and hence all points on it) rotates about a fixed center O, and carries a pin whose axis is at point A. Another point, B, is located on the disk in the position as shown. Having been given or having assumed the velocity of A, it is required to find the velocity of B.

Solution: First Method. Let us assume the velocity of the pin A to be represented by the vector Am. Since the disk is rotating in a counterclockwise direction, as shown in the figure, the vector Am is drawn perpendicular to, and to the left of, OA, the radius of the pin.

With the compass centered at O, the radius OB of point B is now rotated into the radius of A, and thus takes the new position OB_1. Point m is connected to center O by the line Om, forming the right triangle OAm; and line B_1n, drawn perpendicular to OB_1, is then drawn terminating in, or being cut off by, line Om, forming a right triangle OB_1n.

It is evident that the right triangles OAm and OB_1n are similar; therefore in these triangles,

$$\frac{B_1n}{Am}=\frac{OB_1}{OA}$$

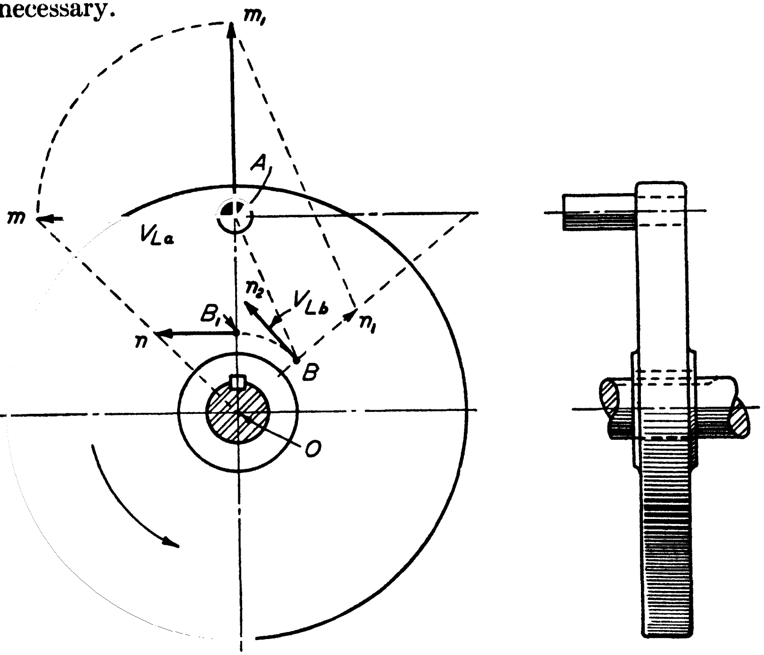
In this equation, OB_1 is the radius of B, OA is the radius of A, and Am is (or represents) the velocity of A, so that

$$\frac{B_1n}{\textit{line representing velocity of } A}=\frac{\textit{the radius of } B}{\textit{the radius of } A}$$

Since formula (12) states that the linear velocities of any two points of the same rotating body are directly proportional to their radii, in the above proportion or equation,

B_1n represents the velocity of B. *Ans.*

Since the radius of B was rotated during the solution of our problem, B_1n is actually in a revolved position in the figure. It may be counter-revolved, if desired, to its proper position, which is perpendicular to OB at B as shown in the figure by the vector Bn_2. Since the magnitude of the velocity is generally all that is required (the direction is obvious), such counter-revolving of the vector is generally unnecessary.



DRAWING SCALE 1" = 12"
VELOCITY SCALE 1" = 1,000 f.p.m.

Fig 42 Disk with Pin

Second Method. The analysis of the problem in this second method is exactly the same as that used in the first method. The solution differs only in the use of similar oblique triangles instead of similar right triangles.

Having assumed the vector Am to represent the velocity of A, it is laid out as before. It is then revolved into the line of the radius of A so that it takes the position as shown by Am_1. For the next step, points A and B are connected by the line AB. This is followed by drawing a line through point m_1, parallel to line AB and intersecting (the line of) the radius of B at n_1. In this manner, two similar oblique triangles, Om_1n_1 and OAB, are formed. In these similar triangles, AB of triangle OAB, being parallel to m_1n_1 of triangle Om_1n_1, cuts the sides Om_1 and On_1 of triangle Om_1n_1 in such a manner that

$$\frac{Bn_1}{Am_1}=\frac{OB}{OA}$$

or

$$\frac{Bn_1}{\textit{line representing velocity of } A}=\frac{\textit{the radius of } B}{\textit{the radius of } A}$$

It is evident from the above statement that

Bn_1 represents the velocity of B. *Ans.*

Therefore the length of Bn_1 should be the same as that of B_1n, the result obtained by the first method. As previously stated, the numerical value, or magnitude, of the velocity of B can be obtained by multiplying the length of its vector by the velocity scale used in laying out the vector Am.

In some of these velocity problems, the velocity of some point of the link, like B, is given instead of the velocity of the pin A, and it is required to find the velocity of A. In such a case, the method of procedure is similar to that of the preceding work. For instance, if such were the case in Fig. 41, a vector, V_{Lb}, representing the known velocity of B, would be drawn to some scale. A line would be drawn then from O through n, the end of V_{Lb}, and extended to cut off Am at m, thus determining the unknown velocity, V_{La}.

Note. The crank of Fig. 41 and the disk of Fig. 42 are such links as might be used as a driver or follower in a four-bar linkage. If either of them were used as, say, the driver A of the linkage represented by the skeleton diagram of Fig. 43, the line of the link that represents A in that figure would be the line OA of Figs. 41 and 42. The instant centers ad and ab of Fig. 43 would be the fixed center O and the center of the pin A, respectively, of Figs. 41 and 42.

Example. Suppose that the length of the resulting vector, representing the velocity of B of the preceding example, is $2/5$ inch. If the velocity scale used in the solution is 1000 feet per minute to the inch, what is the magnitude of the velocity of B?

Solution. Multiplying the length of the vector by the velocity scale,

$$\tfrac{2}{5}\times 1000=400 \text{ f.p.m.} \quad \textit{Ans.}$$

Relative Velocities of Points of a Linkage. The material of the preceding article has demonstrated that the linear velocity of any point of a rotating link can be obtained if the velocity of some point of that link is given. Of course the center of rotation of the link must be known so that the radii of the points can be obtained for use in the solution. If the link has a motion of pure rotation, its instant center about which it rotates with respect to the fixed link or frame is a fixed center and is definitely given; otherwise its instant center can always be located for the given phase by the use of Kennedy's theorem. Thus in Fig. 43 the connector *B* is rotating, with respect to the frame, about the instant center *bd*, at the instant shown by the given phase; and the radius of point *ab*, taken as a point of link *B*, is *bd-ab*; while the radius of point *M*, also considered as a point of link *B*, is *bd-M*.

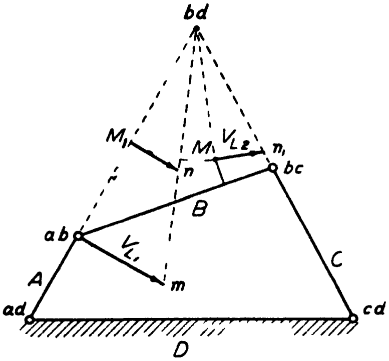


Fig 43

Hence if the velocity of such a point of *B* as *ab* were known, the velocity at the given phase of any other point of *B*, say *M*, could be found by the same (or similar) methods as those employed in Figs. 41 and 42.

It is evident in Fig. 43 that *ab* is the instant center of the relative motion of *A* with respect to *B*, and as such it is the only point which they have in common; that is, it is the only point that can be considered as a point of either of these two links. Since a point can have but one motion at any instant, the motion of *ab* as a point of *B* must be the same as the motion of *ab* as a point of *A*. Hence the velocity of point *ab* can be established by considering it as a point of the driver *A* of the linkage, in which case it has the distance *ad-ab* as a radius. Then this point *ab* is a point of link *B* whose velocity is known. The velocity of any other point of *B*, such as *M*, can then be found.

Since this analysis is entirely general, it follows that if the velocity of a point of any link of a linkage is known (or assumed), the velocity of a point of any other link of that linkage can be obtained. Such a comparison of the velocities of points of two different links is always made through the one and only instant center which they have in common. This is known as their common centro.

Relative Velocities, Four-Bar Linkage, Open Phase. In Figs. 43 and 44 a four-bar linkage is illustrated, with A the driver, B the connector, C the follower, and D the frame. The linear velocity V_{L1} (of the end point ab of the driver) is given, and it is required to obtain velocity V_{L2} (of the point M of the link B) at the given phase. Two methods of solution are given, the first as shown by Fig. 43, the second as shown by Fig. 44.

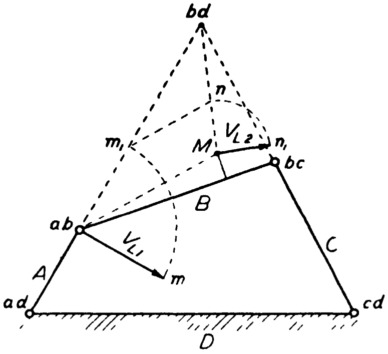


Fig. 44

Note. Since all links of a mechanism derive their motion from the driver, the velocity of some point thereof is always assumed, and the velocities of points of the same or other links can be determined on the basis of this assumed (or known) velocity. Since the methods of solution shown in Figs. 43 and 44 must use the common centro ab, the solution is simplified to some extent by assuming its velocity. In reality it is the angular velocity of the driver that is assumed, for from formula (17) the angular velocity of a rotating body is equal to the linear velocity of any point of the body, divided by the radius of that point. So here the angular velocity V_{a1} of the driver is equal to V_{L1} divided by the radius ad-ab.

Solution: First Method. Since point M, whose velocity is to be determined, is a point of link B, it will be necessary to locate the instant center bd in order that the radius of M, which is bd-M, may be drawn. It is evident from Chapter III that *bd* must lie on line *ad-ab,* as well as on line *cd-bc.* Hence it must be at the intersection of these two lines as shown.

Having drawn the radius *bd-M,* the vector *ab-m* (representing assumed velocity V_{L1} of centro *ab*) is laid off perpendicular to *ad-ab,* and point *m* is connected to *bd,* forming the right triangle *bd-ab-m.*

With *bd* as a center, radius *bd-M* is rotated into the radius of *ab* so that it takes the position $bd\text{-}M_1$. Line M_1n is drawn perpendicular to $bd\text{-}M_1$ terminating in line *bd-m* at point *n* to form the right triangle $bd\text{-}M_1\text{-}n$.

Since the right triangles *bd-ab-m* and $bd\text{-}M_1\text{-}n$ are similar,

$$\frac{bd\text{-}M_1}{bd\text{-}ab}=\frac{M_1n}{ab\text{-}m}$$

In this equation, $bd\text{-}M_1$ and *bd-ab* are the radii of points *M* and *ab,* respectively; for *M* and *ab* are both points of the link *B,* which rotates (with respect to the frame) about the center *bd.* Also *ab-m* by construction is equal to V_{L1}.

Therefore, from the preceding equation,

$$\frac{\textit{the radius of } M}{\textit{the radius of } ab}=\frac{M_1n}{V_{L1}}$$

so that, at this phase, M_1n represents the velocity of point *M* to the same scale as that used for V_{L1}. See formula (12). The velocity of *M* is perpendicular to its radius, so the true position of V_{L2} is represented by the vector Mn_1, drawn perpendicular to *bd-M* and directed to the right of it as shown. In most cases the direction of the velocity is easily obtained by noting the direction of rotation of the driver and closely inspecting the layout of the linkage. (See the article on Directional Relation in Chapter III.)

Second Method. It should again be noted that the linear velocity of the instant center *ab* is given, and that this point is one which can be treated as either a point of link *A* or as a point of link *B.* As before, vector *ab-m,* or V_{L1} (the velocity of *ab*) is laid out perpendicular to the line of the link *A*.

Having located the instant center *bd* (Fig. 44) about which all points of *B* rotate at the given instant, the radius of *M*, which is *bd-M,* is drawn. A line *M-ab* is now drawn from *M* to the common centro *ab.*

The vector *ab-m,* or V_{L1}, is revolved into the radius of *ab* (the line of the link *A*) taking the position $ab\text{-}m_1$; and a line $m_1\text{-}n$ is drawn through m_1, parallel to the line *M-ab* and terminating in the radius of M at point n.

Two similar oblique triangles, n-*bd*-m_1 and *M-bd-ab*, are thus formed, in which side m_1n of the former is parallel to side *ab-M* of the latter, so that

$$\frac{bd\text{-}M}{bd\text{-}ab} = \frac{Mn}{ab\text{-}m_1} = \frac{Mn}{ab\text{-}m}$$

In this equation the line *bd-M* is the radius of M, the line *bd-ab* is the radius of *ab*, and *ab-m* is V_{L1}. Hence

$$\frac{\textit{the radius of } M}{\textit{the radius of } ab} = \frac{Mn}{V_{L1}}$$

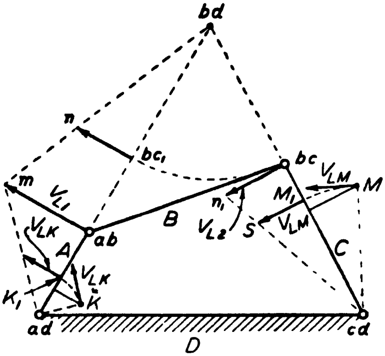


Fig 45

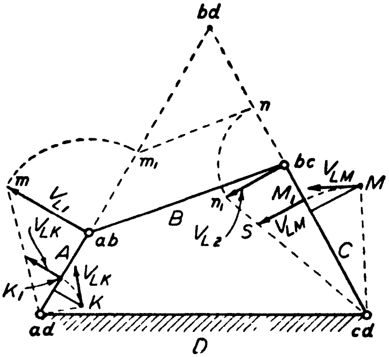


Fig. 46

Therefore, from formula (12), Mn must be the velocity of point M to the scale used in laying out V_{L1}. Since the driver A is shown by V_{L1} to rotate in a clockwise direction, Mn may be rotated clockwise to the position Mn_1 which is perpendicular to *bd-M*. Mn_1 is the vector representing the required velocity, V_{L2}, of point M of link B at the phase or instant given by the relative positions which the links occupy in the figure. Consequently V_{L2} is the Instantaneous Velocity of point M.

Another example dealing with relative velocities at a given phase of a four-bar linkage is shown in Figs. 45 and 46. The linkage used is the same one that was used in the preceding example of Figs. 43 and 44, but in this case the velocity which is assumed is that of a point K of the driver A, and the velocity which is to be found is that of point M of the follower C. It will be noted that neither K nor M is on the line of its respective link. Two methods of solution are presented, which follow as closely as possible the methods used in the preceding example. The following analysis serves for both methods.

Analysis. In any linkage, the velocities of the common centros can be obtained in order or sequence, passing from the first link or driver (whose velocity is known or assumed) through the successive links to the follower. This is accomplished by considering each common centro, first as a point of one link in order to obtain its velocity and then as a point of the other link (to which it is common) in order to use this velocity in determining the velocity of another centro (or point) of the second link.

Thus in the example of Figs. 45 and 46, the common centro *ab* (of links *A* and *B*) is considered, to begin with, as a point of the driver *A*, with a radius *ad-ab*. Its velocity can be obtained on the basis of the given velocity of point *K*, which is another point of the driver. Having found the velocity of *ab*, it is treated as a point of link *B*, with a radius *bd-ab*, and so the velocity is obtained of common centro *bc*, another point of *B*. Since *bc* is also a point of the follower *C*, the velocity of any point of *C*, such as point *M*, can then be determined. It should be noted that when *bc* is taken as a point of link *B*, its radius is *bd-bc*, but when it is taken as a point of the follower *C*, its radius is *cd-bc*, for the motion of all links is taken relative to the frame.

First Method. See Fig. 45.

Step 1. Preliminary work. Having laid out the phase of the linkage under consideration, name all the links and obtain the instant centers that are suggested by the analysis of the problem as necessary. Most of these instant centers can be obtained by inspection of the layout of the phase.

Assume the vector V_{Lk} to represent the velocity of the given point *K* of the driver *A*. This vector is drawn perpendicular to the radius *ad-K*.

Step 2. To obtain V_{L1}, the velocity of the common centro *ab* of links *A* and *B*:

The common centro *ab* is considered here as a point of *A*; therefore its radius is *ad-ab*.

With *ad* as a center and *ad-K* as a radius, revolve *K* into the radius of *ab*, where it takes the position K_1. Lay off the vector V_{Lk} perpendicular to *ad*-K_1 at K_1. This is in reality a revolved position of V_{Lk}.

Draw a line from *ad* through the end of the revolved position of V_{Lk} and extend it until it intersects, at point *m*, a line *ab-m*, which is drawn perpendicular to *ad-ab*.

ab-m is V_{L1}, the velocity of the common centro *ab*. Why?

Step 3. To obtain V_{L2}, the velocity of the common centro *bc*:

ab and *bc* are considered here as points of link *B*, and since all points of *B* rotate relative to the frame at this instant about *bd*, their radii are *bd-ab* and *bd-bc* respectively.

With *bd* as a center, rotate the radius *bd-bc* until it coincides with *bd-ab*, point *bc* in its revolved position being called bc_1.

Draw line *bd-m*, and at bc_1 erect a perpendicular to *bd-ab* intersecting *bd-m* at *n*.

Then bc_1-*n* represents the velocity of *bc*. Why?

Draw *bc*-n_1 perpendicular to *bd-bc* (and hence perpendicular to *cd-bc*) and equal in length to bc_1-*n*. Why is *bc*-n_1 directed to the left?

bc-n_1 is the true position of V_{L2}, the velocity of the common centro *bc*.

Step 4. To obtain V_{Lm}, the required velocity of point *M* of link *C*:

With *cd* as a center, rotate the radius of *M*, which is *cd-M*, into the radius of *bc* (the line of the link *C*) so that *M* in its revolved position is at M_1.

Draw a line from *cd* to n_1 and at M_1 erect a perpendicular to the line of the link *C*, intersecting *cd*-n_1 at *S*.

Then M_1S represents V_{Lm}. Why?

Draw V_{Lm} in its true position perpendicular to *M-cd* at *M* and equal in length to M_1S.

Second Method. Refer to Fig. 46.

Steps 1 and 2. Same as in the first method.

Step 3. To obtain V_{L2}, the velocity of the common centro *bc*:

Rotate V_{L1} (*ab-m*) into the radius of *ab*, so that *m* falls at m_1.

Draw line m_1n parallel to the line *ab-bc* (the line of the link *B*) intersecting *bd-bc* at *n*.

bc-n represents the velocity of *bc*. Why?

Revolve *bc-n* about *bc* as a center until it is perpendicular to *bc-cd* and takes the position *bc*-n_1, which is the true position of V_{L2}, the velocity of *bc*.

Step 4. To obtain V_{Lm}, the velocity of point *M* of link *C*:

Proceed exactly as in Step 4 of the first method.

Note. The student should notice that the velocity V_{L2} of bc in Figs. 45 and 46 (or the velocity of M in Figs. 43 and 44) varies directly as the velocity V_{L1} of point ab; that is, if the vector ab-m, representing V_{L1}, were made twice as long as in the figure, then the length of V_{L2} would likewise turn out to be twice its present length. Therefore the ratio of $\frac{V_{L1}}{V_{L2}}$ is independent of the value given to V_{L1} and is dependent only on the relative lengths of links in the mechanism and their relative placement in the particular phase being investigated. When the ratio of the linear velocities $\frac{V_{L1}}{V_{L2}}$ is all that is required at a given phase, the linear velocity of the driver or its corresponding angular velocity need not be known numerically but can be assumed. This is the reason for implying in the statement of an example that if the linear velocity of some point of the driver is not known it may be assumed. Later on in this chapter, the linear velocity V_{L2}, or the ratio $\frac{V_{L1}}{V_{L2}}$, will be required at successive phases throughout a cycle of a mechanism. In such case it is evident that the driver is assumed to have a constant angular velocity and hence whatever velocity is assumed for V_{L1}, it must then be used at every phase of the cycle.

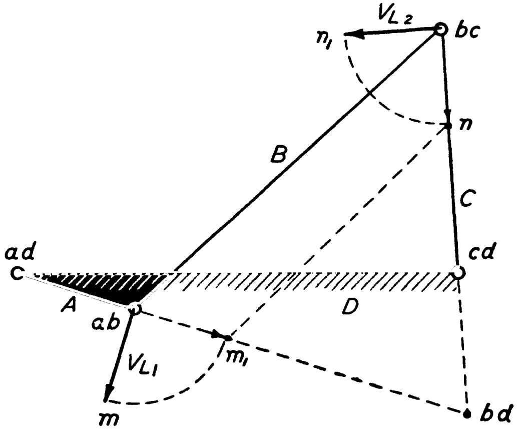


Fig. 47

Relative Velocities, Four-Bar Linkage, Crossed Phase. The crank A of the linkage illustrated in Fig. 47 has reached a position in which the connector B crosses the line of the frame D at a point between the permanent centers ad and cd. It is evident from the statement made in the article dealing with Directional Relation of Driver and Follower in Chapter III that if the driver is rotating in a clockwise direction as shown in the figure, the follower, C, is rotating in the opposite, or in a counterclockwise, direction at this phase.

In this linkage the angular velocity of the driver A (and hence of all points of the driver) is given or known; and V_{L1}, the linear velocity of its pin at the instant center *ab*, is found by multiplying the angular velocity by the radius *ad-ab*. See formula (16). It is then laid out to some scale on the diagram as the vector *ab-m*. It is required to obtain the velocity of the pin *bc* at the end of the follower C.

Solution.

Step 1. Preliminary work.

After laying out the given phase under consideration, name all the links and obtain all necessary instant centers.

Locate the vector *ab-m* to represent V_{L1}, the velocity of the pin or instant center *ab* which is the common centro of links A and B. Of course *ab-m* is perpendicular to *ad-ab*, the radius of *ab* as a point of link A. It is also directed along this perpendicular so as to indicate the direction of rotation of the driver.

Step 2. To obtain the velocity V_{L2}:

Rotate *ab-m* about *ab* as a center until it takes the position *ab-*m_1, in which it lies along the line of the link A.

Note. In any of these problems, *ab-m* can be rotated either clockwise or counterclockwise about the center *ab*, until it is coincident with the line of the link A.

Through m_1 draw line m_1n parallel to the position of the connector B at this phase and intersecting the line of the link C at point n.

Then *bc-n* represents V_{L2}, the velocity of point *bc*, to the same scale as that used in laying out the vector *ab-m*.

For the true position of V_{L2}, rotate *bc-n* about *bc* to the position *bc-*n_1, in which it is at right angles to the radius *cd-bc* of *bc*, and so directed as to indicate that the follower C is rotating counterclockwise about its fixed center *cd*.

Relative Velocities, Slider-Crank Mechanism. In the slider-crank mechanism of Fig. 48, A is the driver or crank; B, the connecting rod; C, the slider; and D, the frame. The angular velocity of the crank in radians per minute, or the corresponding r.p.m. of the crank, is such that it produces a linear velocity V_{L1}, of the crank pin *ab*, which is represented by the given vector *ab-m*. It is required to obtain the velocity of the slider.

Analysis: The motion of the slider C is one of translation back and forth along a straight line. Therefore the velocities of all points of the slider are equal at any given phase or instant, and hence the velocity of the slider is this common velocity of its points.

In order to obtain the velocity of the slider by the method of solution which is shown in Fig. 48, that point of it is used whose velocity can be most directly and hence most easily secured. It is evident that this point is the common centro *bc* of the links *B* and *C*. This point *bc* is selected because it can be considered as a point of the connector *B*. Its velocity as a point of *B* can be obtained in a very direct manner from the known velocity of another point, *ab* of the same link *B*, for the velocities of these two points are directly proportional to their radii. These radii are made known, and hence usable, in the graphical solution by locating the instant center *bd*, about which all points of the link *B* rotate with respect to the frame at the given instant.

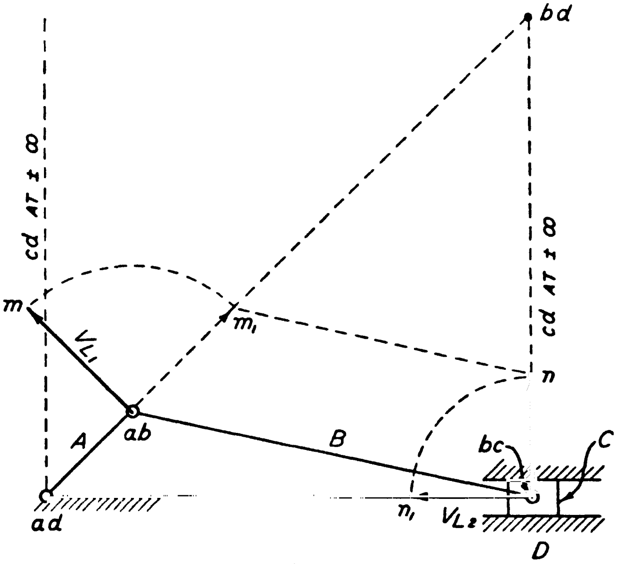


Fig. 48

Solution. Lay out the skeleton diagram of the linkage and name the links.

Secure at least those instant centers which must be available for use in the graphical solution.

Draw *ab-m* to some scale to represent the linear velocity of *ab*. It is of course drawn perpendicular to its radius *ad-ab*.

Rotate *ab-m* into the position *ab-*m_1.

Draw line m_1n through point m_1, parallel to the position of the connector *B* at this phase, intersecting *bd-bc* at point *n*.

The distance *bc-n* represents, or is a measure of, the velocity of *bc,* to the same scale as that used in *ab-m.* It therefore represents V_{L2}, the velocity of the slider or any point thereof.

For the true position of V_{L2}, rotate *bc-n* about *bc* as a center, to the position *bc-*n_1, which is along the straight line path of point *bc*, and directed to the left because the crank is rotating counterclockwise.

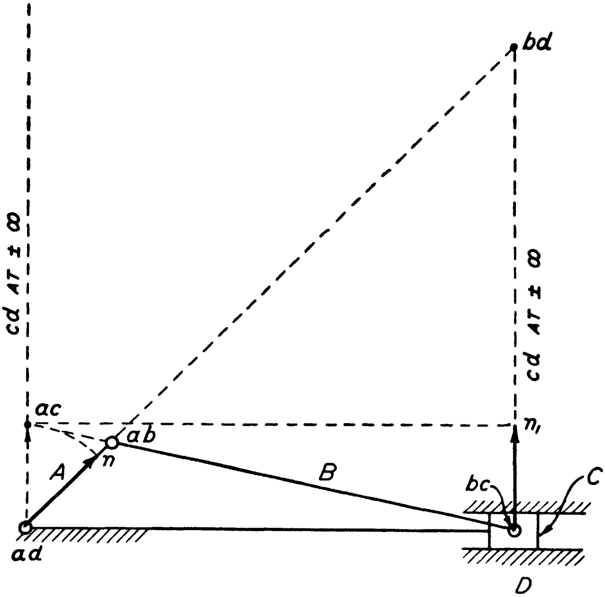


Fig. 49

Another method of solution of the preceding example is shown in Fig. 49. It is based on the assumption that the linear velocity of the crank pin *ab* is represented by the length of the crank, which is the distance *ad-ab.* This method can be used only when this assumption is made.

It will be noted in the following explanation that this method makes use of the instant center *ac,* which played no part in the preceding solution.

Solution. Lay out the skeleton diagram, name the links, and secure all the instant centers. (See Fig. 33.)

In locating the instant center *ac*, the line of the link *B* (extended at this phase) cuts off the segment *ad-ac* from the line which connects *ad* and *cd* (at ∞).

This segment *ad-ac* represents V_{L2}, the velocity of the slider, when the length of the crank represents the velocity of the crank pin.

The line *ad-ac* can be revolved into the line of the driver about *ad* as a center, point *ac* falling at *n* as shown in Fig. 49. This yields a very direct comparison of the velocities of the slider and crank pin; *ad-n* representing that of the slider; *ad-ab,* that of the crank pin.

If it is desired to have the velocity of the slider laid off along the line *bd-bc,* a line can be drawn through *ac* parallel to the line of the frame, cutting *bd-bc* at n_1, so that *bc*-n_1 also represents the velocity of *bc* on the slider.

Proof: An inspection of Fig. 49 will show that triangles *ad-ab-ac* and *bd-ab-bc* are similar. Since in similar triangles the corresponding sides are proportional,

$$\frac{ad\text{-}ac}{ad\text{-}ab}=\frac{bd\text{-}bc}{bd\text{-}ab}$$

In the above proportion or equation, *bd-bc* and *bd-ab* are the radii of points *bc* and *ab*, respectively, when these points are taken as points of link *B*, and hence at this instant they rotate (with respect to the frame) about *bd.* Also *ad-ab* represents V_{L1}, the velocity of *ab;* or *ad-ab* multiplied by the velocity scale is equal to the velocity V_{L1}. Hence from the given equation,

$$\frac{(ad\text{-}ac)(\textit{the velocity scale})}{(ad\text{-}ab)(\textit{the velocity scale})}=\frac{(ad\text{-}ac)(\textit{the velocity scale})}{V_{L1}}$$
$$=\frac{\textit{the radius of bc}}{\textit{the radius of ab}}$$

Therefore from formula (12), (*ad-ac*)×the velocity scale = the velocity V_{L2} of *bc*, or of the slider *C;* or *ad-ac* represents V_{L2}, the velocity of *bc*, and hence of the slider *C*, at the given phase.

Note. The required velocity, that has been obtained in each of these graphical solutions dealing with the four-bar linkage and the slider-crank mechanism, is the instantaneous velocity of the point, or the velocity which the point has at the phase under consideration. At any other phase a different result would be obtained in all probability, even though the motion of the driver were uniform.

Example. The crank of the steam engine mechanism of Fig. 48 is 6 inches in length and makes 200 revolutions per minute. The vectors *ab-m* and *bc-n*, which represent the velocities of the crank pin and piston (slider), are ¾ inch and ⅝ inch in length respectively. It is required to find

(*a*) the linear velocity of the crank pin in feet per minute
(*b*) the angular velocity of the crank pin in radians per minute
(*c*) the angular velocity of the crank in radians per minute
(*d*) the velocity scale used in the graphical solution
(*e*) the linear velocity of the piston in feet per minute.

Solution.

(*a*) Here $R=6$ in. $=\frac{1}{2}$ ft. and $N=200$ r.p.m. Evaluating in formula (11)

$$\begin{aligned} V_L &= 2\pi RN \\ &= 2\times\pi\times\tfrac{1}{2}\times 200 \\ &= 200\pi = 628.32 \text{ f.p.m.} \quad \textit{Ans.} \end{aligned}$$

(*b*) Here $N=200$ r.p.m. Evaluating in formula (14)

$$\begin{aligned} V_a &= 2\,\pi N \\ &= 2\times\pi\times 200 \\ &= 400\pi = 1256.64 \text{ radians per min.} \quad \textit{Ans.} \end{aligned}$$

(*c*) Since the angular velocity of a body is the same as that of every point of the body,

V_a, for the crank $=1256.64$ radians per min. *Ans.*

(*d*) The length of $ab\text{-}m=\frac{3}{4}$ in. and V_L of crank pin $=628.32$ f.p.m. Therefore

$$\begin{aligned} \tfrac{3}{4}\text{ in.} &= 628.32 \text{ f.p.m.} \\ \tfrac{1}{4}\text{ in.} = \tfrac{1}{3}\text{ of }\tfrac{3}{4}\text{ in.} &= \tfrac{1}{3}\text{ of } 628.32 \text{ f.p.m.} \\ &= 209.44 \text{ f.p.m.} \\ 1\text{ in.} = 4\times\tfrac{1}{4}\text{ in.} &= 4\times 209.44 \text{ f.p.m.} \\ &= 837.76 \text{ f.p.m.} \quad \textit{Ans.} \end{aligned}$$

(*e*) From part (*d*) the velocity scale is 1 in. $=837.76$ f.p.m. The length of the vector which represents V_{L2} of Fig. 48 will then be equal to:

$$\tfrac{5}{8}\times 837.76 = 523.6 \text{ f.p.m.} \quad \textit{Ans.}$$

Ratio of Angular Velocities of Driver and Follower. Referring to Fig. 46, let

V_{a1} = the angular velocity of the driver A
V_{a2} = the angular velocity of the follower C

Since V_{L1} is the linear velocity of the instant center *ab*, whose radius as a point of the driver A is *ad-ab*, from formula (17)

$$V_{a1}=\frac{V_{L1}}{ad\text{-}ab} \tag{a}$$

In the same manner, with V_{L2} and *cd-bc*, the linear velocity and radius respectively of the instant center *bc* of the follower C,

$$V_{a2}=\frac{V_{L2}}{cd\text{-}bc} \tag{b}$$

Dividing the first member of step (*a*) by the first member of step (*b*) and equating this quotient to that obtained through dividing the second member of step (*a*) by the second member of step (*b*),

$$\frac{V_{a1}}{V_{a2}}=\frac{\dfrac{V_{L1}}{ad\text{-}ab}}{\dfrac{V_{L2}}{cd\text{-}bc}} \tag{c}$$

from which

$$\frac{V_{a1}}{V_{a2}}=\frac{V_{L1}}{ad\text{-}ab}\times\frac{cd\text{-}bc}{V_{L2}}=\frac{V_{L1}}{V_{L2}}\times\frac{cd\text{-}bc}{ad\text{-}ab} \tag{19}$$

An inspection of formula (19) shows that the angular velocity ratio, $\frac{V_{a1}}{V_{a2}}$, is dependent upon the linear velocity ratio, $\frac{V_{L1}}{V_{L2}}$, and the ratio $\frac{cd\text{-}bc}{ad\text{-}ab}$. The latter will be recognized as the ratio of the length of the follower to the length of the crank, which is a constant for a given four-bar linkage. As stated previously, the ratio $\frac{V_{L1}}{V_{L2}}$ is independent of the actual velocity, V_{L1}, of the end of the driver. Therefore the angular velocity ratio of driver to follower at any phase of a given mechanism can be obtained by assuming the angular velocity of the driver, or the linear velocity of any point thereof.

It is seen in Fig. 46 that *ab* and *bc* are points of the connector B which rotates at the given phase about the center *bd*. The radii of *ab* and *bc* as points of B are *bd-ab* and *bd-bc* respectively. Hence from formula (12)

$$\frac{V_{L1}}{V_{L2}}=\frac{bd\text{-}ab}{bd\text{-}bc}$$

Substituting this value of $\frac{V_{L1}}{V_{L2}}$ in formula (19),

$$\frac{V_{a1}}{V_{a2}}=\frac{bd\text{-}ab}{bd\text{-}bc}\times\frac{cd\text{-}bc}{ad\text{-}ab} \tag{20}$$

The above expression for the angular velocity ratio should further clarify the statement that $\frac{V_{L1}}{V_{L2}}$ (and therefore $\frac{V_{a1}}{V_{a2}}$) is independent of the actual value of V_{L1}; for since $\frac{V_{L1}}{V_{L2}}$ is equal to the ratio of the radii *bd-ab* and *bd-bc*, it is dependent only on the relative lengths of the links and their placement at a given phase. With *bd* taking different positions at different phases, it is evident that $\frac{V_{L1}}{V_{L2}}$ as well as $\frac{V_{a1}}{V_{a2}}$ will vary in general, even though the angular velocity of the driver is constant throughout a cycle.

Angular Velocity Theorem. When the angular velocity ratio of driver to follower is obtained through the use of formula (20), four lines must be measured. Another formula by which this ratio may be obtained at any phase, and in which only two lines need be measured, will now be derived.

A skeleton diagram of a four-bar linkage is given in Fig. 50 to illustrate this derivation, and all of the instant centers are located thereon. Of these instant centers, one in particular is of great utility in this derivation. It is the instant center *ac*, which is the common centro of the driver *A* and the follower *C*.

It is evident from Kennedy's theorem that *ac* must fall on the line *ab-bc* which is the line of the connector *B;* for this line is given or determined by two of the three instant centers of the three links *A*, *B*, and *C*, whose third instant center is *ac*. The latter must also fall on line *ad-cd*, which is the line of the frame sometimes referred to as the line of fixed centers; for this line is determined by two of the three instant centers of the three links *A*, *C*, and *D*, whose third centro is also *ac*. Hence *ac* must be on both the line of the frame and the line of the connector; therefore it must be at their intersection as shown in the figure.

Now *ac*, being the common centro of *A* and *C*, is the only common point that these links possess. As a point of *A*, the radius of *ac* is seen to be *ad-ac;* and as a point of *C*, its radius is seen to be *cd-ac*. It should be noted in particular that these radii are also the segments into which the line of the fixed centers is cut by the line of the connector; they will always be such segments, for *ac* at all phases throughout the cycle of the mechanism is somewhere along the line of the fixed centers.

Derivation:

Assume the angular velocity of the driver *A* by taking *ab-m* perpendicular to *ad-ab* as the linear velocity of its point *ab*.

Rotate *ad-ab* into the line of the frame so that *ab* falls at ab_1.

Draw ab_1-m_1 perpendicular to *ad*-ab_1 and equal in length to *ab-m*.

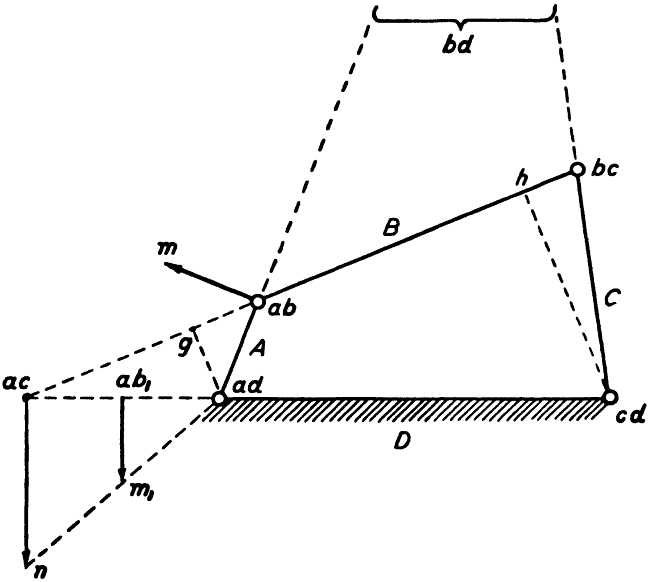


Fig. 50

Draw *ac-n* perpendicular to *ad-ac* and intersecting, at point *n*, a line drawn from *ad* to m_1.

In the similar right triangles *ad*-ab_1-m_1 and *ad-ac-n*,

$$\frac{ac\text{-}n}{ab_1\text{-}m_1} \text{ or } \frac{(\ ac\text{-}n\ )\ (\textit{the velocity scale})}{(ab_1\text{-}m_1)\ (\textit{the velocity scale})} = \frac{ad\text{-}ac}{ad\text{-}ab_1}$$

or

$$\frac{(ac\text{-}n)\ (\textit{the velocity scale})}{\textit{the velocity of ab}} = \frac{\textit{the radius of ac}}{\textit{the radius of ab}}$$

Therefore (*ac-n*) multiplied by the velocity scale = the velocity of *ac*; or *ac-n* represents graphically the velocity of *ac* to the same scale as that used in laying out *ab-m*.

Since from formula (17) the angular velocity V_{a1}, of the driver A, is equal to the linear velocity of its point *ac*, divided by the radius *ad-ac* of that point as a point of A,

$$V_{a1}=\frac{(ac\text{-}n)\ (the\ velocity\ scale)}{ad\text{-}ac} \tag{a}$$

Likewise the angular velocity V_{a2}, of the follower C, is equal to the linear velocity of its point *ac*, divided by the radius *cd-ac* of that point as a point of C. Therefore

$$V_{a2}=\frac{(ac\text{-}n)\ (the\ velocity\ scale)}{cd\text{-}ac} \tag{b}$$

From steps (*a*) and (*b*),

$$\frac{V_{a1}}{V_{a2}}=\frac{\dfrac{(ac\text{-}n)\ (the\ velocity\ scale)}{ad\text{-}ac}}{\dfrac{(ac\text{-}n)\ (the\ velocity\ scale)}{cd\text{-}ac}}$$

$$\frac{V_{a1}}{V_{a2}}=\frac{(ac\text{-}n)\ (velocity\ scale)}{ad\text{-}ac}\times\frac{cd\text{-}ac}{(ac\text{-}n)\ (velocity\ scale)}$$

Canceling like factors in the second member,

$$\frac{V_{a1}}{V_{a2}}=\frac{cd\text{-}ac}{ad\text{-}ac} \tag{21}$$

Since these radii *cd-ac* and *ad-ac* are the segments into which the line of fixed centers of the mechanism is cut by the connector B, formula (21) may be stated as follows:

In any four-bar linkage, at any phase, the angular velocities of driver and follower are to each other inversely as the segments into which the line of fixed centers is cut by the connector.

In an open phase of a four-bar linkage like that shown in Fig. 50, point *ac* is always on the extension of the line of fixed centers. In such a case the line of fixed centers is said to be cut or divided externally into segments by the line of the connector.

In a crossed phase of a four-bar linkage as in Fig. 47, the point *ac* is always located somewhere between the fixed centers; and the line of fixed centers is said to be cut or divided internally into segments by the line of the connector.

In order to proceed further in this investigation, referring again to Fig. 50, draw lines *ad-g* and *cd-h* from the fixed centers *ad* and *cd*, respectively, to the line of the connector *B*. By so doing, two similar right triangles, *ad-g-ac* and *cd-h-ac*, are formed, in which

$$\frac{cd\text{-}ac}{ad\text{-}ac}=\frac{cd\text{-}h}{ad\text{-}g}$$

so that if $\frac{cd\text{-}h}{ad\text{-}g}$ is substituted in formula (21) for $\frac{cd\text{-}ac}{ad\text{-}ac}$

$$\frac{V_{a1}}{V_{a2}}=\frac{cd\text{-}h}{ad\text{-}g} \tag{22}$$

Formula (22) may be stated as follows:

In any four-bar linkage, at any phase, the angular velocities of driver and follower are to each other inversely as the perpendiculars which are drawn from their fixed centers to the line of the connector.

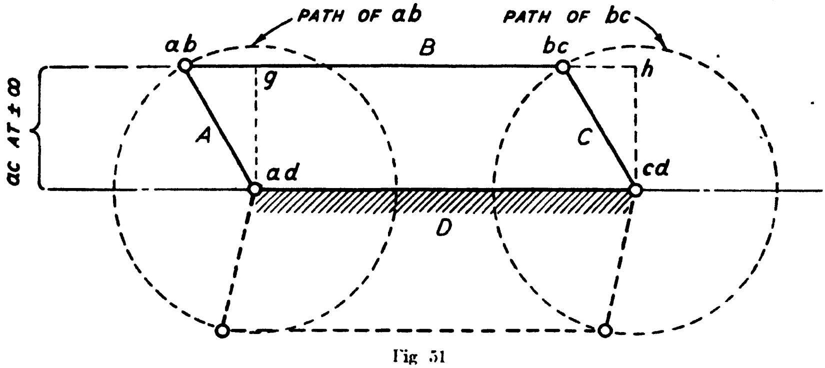


Fig. 51

The statements of formulas (21) and (22) combine to form a statement which is known as the Angular Velocity Theorem.

Constant Angular Velocity Ratio. Should point *ac* have the same position (or be a fixed point) on the line of fixed centers at every phase of a cycle of the linkage, it is evident that the ratio of the segments *cd-ac* and *ad-ac* would be constant, for neither of them would change in length through successive phases. Since points *ad* and *cd* are fixed on this line of centers, no two different positions of *ac* can possibly produce segments whose ratios are the same.

Therefore the condition for a Constant Angular Velocity Ratio between driver and follower is that their common centro must be in a fixed position on the line of fixed centers throughout the cycle.

This condition is met with in the four-bar linkage when the driver and follower are of the same length and the length of the connector is equal to the distance between the fixed centers. The skeleton diagram of such a linkage is given in Fig. 51. Two phases of the linkage are shown therein. From this figure it will be noted that the lengths of the perpendiculars *ad-g* and *cd-h* will be equal to each other at every phase. This will give an angular velocity ratio that will be equal to unity at every phase, and hence constant. Also the condition of constant angular velocity ratio is met with, for the common centro *ac* is always at infinity on the line of fixed centers.

A practical application of the linkage of Fig. 51 is illustrated in the parallel rod connecting the drivers of a locomotive as shown in Fig. 52. The axles of the drivers are the fixed centers *ad* and *cd*, the frame of the locomotive forms the fixed link *D*, the drivers themselves form the links *A* and *C*, and the parallel rod is the intermediate rigid link connector *B*. Here is the case where the frame of a machine is considered as the fixed or stationary link, even though it has relative motion with the ground.

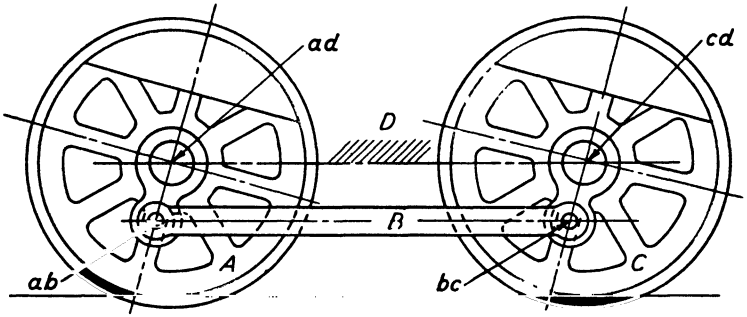


Fig 52

Influence of Length of Links in a Four-Bar Linkage. Through proportioning the links of a four-bar linkage in various ways, the following conversions of motion can be effected between driver and follower:

(*a*) Rotation of the driver is converted into rotation of the follower with a constant angular velocity ratio existing between them.

(*b*) Rotation of the driver is converted into rotation of the follower with a variable angular velocity ratio existing between them.

(*c*) Rotation of the driver is converted into oscillation (rotative reciprocation) of the follower.

(*d*) Oscillation of the driver is converted into oscillation of the follower.

(*e*) Rotation of the driver is converted into (rectilinear) reciprocation of the follower.

The main requirements in proportioning the links of each of the above types are as follows:

(*a*) This type is shown in Fig. 51. The relative lengths of its links must be such that the length of A is equal to the length of C, and the length of B must be equal to the length of D; that is

$$A = C \text{ and } B = D$$

Besides the parallel rod mechanism of a locomotive, another practical construction using the principle of this type is the parallel ruler of the drafting room.

(*b*) The linkwork of this type is generally spoken of as a drag-link mechanism. If the driver has a uniform rotation, it imparts to its follower a highly variable angular velocity as driver and follower make complete revolutions about their fixed centers. Such a linkage is shown in Figs. 6, 8, and 36. In this type the links must be so proportioned that the length of B is greater than the result obtained by subtracting the length of A from the sum of the lengths of C and D; also the length of B must be less than the result obtained by subtracting the length of D from the sum of the lengths of A and C; that is

$$B > C + D - A$$
$$B < A + C - D$$

(*c*) This type is the one used in Figs. 27, 29, and 62. Referring to both Figs. 29 and 62, it may be seen that as the driver A rotates about its fixed center, the follower C travels back and forth, or oscillates, between its limiting positions C_1 and C_2 of Fig. 29. When A has a constant angular velocity, the linear velocities are shown by Fig. 62 to vary greatly; hence the angular velocity of C is extremely variable. The conditions which must be met in type (*c*) are as follows:

$$B - A > D - C$$

and $$B + A < D + C$$

In order to show how to find or locate the phases that will give the limiting positions C_1 and C_2 (of the follower C) of Figs. 29 and 62 (in which positions the instant center *bc* is at *n* and *m*, respectively) Fig. 53, which is a layout of the same linkage, should be examined. From this figure it is evident that when *bc* is at point *n* along its path (the circular arc *mn*) the driver A and the connector B are in the same straight line and it is impossible for C to rotate further to the right as A rotates in either direction about *ad*. Hence *n* is at a distance from the fixed center *ad*, equal to the sum of the lengths of A and B. With the sum of these lengths as a radius, and point *ad* as a center, a circular arc may be described which will intersect the path of *bc* (which is centered at *cd*) at *n*. Point *n* is connected to point *ad* by a straight line. This line is position A_1 of the driver A, from *ad* to the point s_1

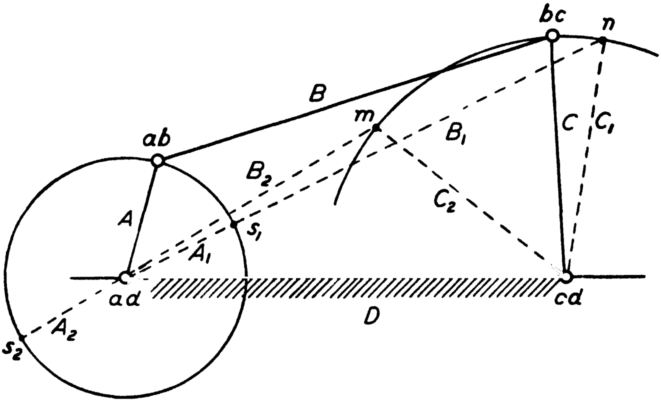


Fig 53

where it intersects the path of *ab*; and from there on it is the position B_1 of the connector B. To obtain the corresponding position, C_1, of the follower C, *n* is connected to the fixed center *cd*. Hence this extreme or limiting position of C is given in Fig. 53 by the phase which is made up of the dotted lines A_1, B_1, C_1, and D.

When *bc* is at point *m*, and the follower C has taken the position C_2, links A and B are again in the same straight line; but, in taking such a position, B overlaps A; hence *m* is at a distance from *ad* which is equal to the length of B minus the length of A. With this distance as a radius and *ad* as a center, an arc is described which intersects or cuts the path of *bc* at *m*. Now if a line is drawn from *m* to *ad* and is extended beyond *ad* to intersect the circular path of *ab* at s_2, *ad*-s_2 and s_2m are the positions of A and B respectively when C is at C_2. An inspection of the figure shows that as A continues its rotation about its fixed center, point *bc* cannot travel to the left beyond the point *m*.

Assuming that A rotates clockwise, its rotation from position A_1 to position A_2 will cause C to rotate from C_1 to C_2, and its rotation from A_2 to A_1 will cause C to return from C_2 to C_1. Hence as A rotates about *ad*, C oscillates between C_1 and C_2, with *cd* as a center.

(*d*) The relative lengths of the links of Fig. 54 are such that neither the driver nor the follower can make a complete turn about its fixed center. In such a linkage, however, both driver and follower can oscillate. The links are proportioned in this case so that

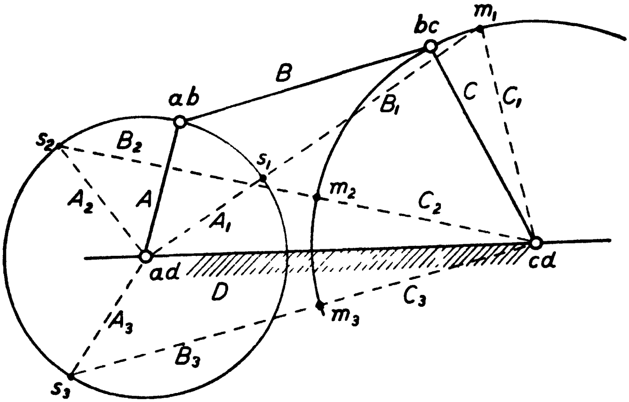


Fig. 54

$$B+C<A+D$$

and

$$B-A<D-C$$

With such proportioning, the extent of the motion of the driver and follower is dependent upon the definite values assigned to the links. For instance, with the lengths assigned to the links of Fig. 54, the driver can move from the position A_2 through A_1 to the position A_3 and, by so doing, will cause the follower to move from the position C_2 to C_1 and thence back through C_2 to the position C_3. As the driver returns from A_3 through A_1 to A_2, the follower will be caused to move from C_3 to its limiting position C_1 and thence back to C_2.

To find the limiting phases (C_1 of the follower and A_2 and A_3 of the driver) of the mechanism of Fig. 54, the procedure is as follows:

With *ad* as a center and the length of A as a radius, describe the circle along which *ab* will move.

With *cd* as a center and the length of C as a radius, describe the circle along which *bc* will move.

With *ad* as a center and with a radius equal to the sum of the lengths of A and B, describe an arc intersecting the path of *bc* at m_1.

Note. Although an arc so described would cut the circle (along which the circular path of *bc* is located) at a second point on the opposite side of the circle, it is evident that with the links placed as shown, point *bc* can travel no farther than m_1 and hence this second intersection is of no consequence.

Connect *ad* and m_1. This line will be position A_1 of the driver from *ad* to s_1, and it will be the corresponding position, B_1, of the follower from s_1 to m_1. Connect m_1 to the fixed center *cd* for C_1, the position of the follower at this phase.

Again using the sum of the lengths of A and B as a radius, but with a center at *cd*, describe an arc intersecting the circle of the path of *ab* at s_2 and s_3. Draw line *cd*-s_2 intersecting the circular path of *bc* at m_2. Also draw *ad*-s_2. Then *ad*-s_2 is a limiting position, A_2, of the driver, and the corresponding positions of the follower and connector at this phase are *cd*-m_2 (or C_2) and m_2s_2 (or B_2), respectively. Draw line *cd*-s_3 intersecting the circular path of *bc* at m_3. Also draw *ad*-s_3. The latter is the limiting position A_3 of the driver, and *cd*-m_3 and m_3s_3 are the corresponding positions C_3 and B_3 of the follower and connector respectively. C_3 is a limiting position of the follower C.

(*e*) Rotation of the driver can be changed into reciprocation of the follower by using a sliding block for the follower instead of a crank or lever. By this change in the construction of the follower, the resulting four-bar linkage is a slider-crank mechanism such as shown in Figs. 48, 59, and 60. Should the crank of this mechanism have a constant angular velocity, the linear velocity of the slider will be highly variable.

Note. In all four-bar linkages the follower can serve as the driver, so that conversions of motion as listed can be effected in the reverse order. For instance, in type (*e*) the slider with its reciprocation may be the driver, and then reciprocation is converted by the mechanism into rotation. In fact this is what occurs in all reciprocating engines.

Dead Point. It is obvious in the four-bar linkage that the driver compels motion of the follower through introducing some component of its motion into, and hence along the line of, the connector. Now it happens that there are certain phases of some linkages in which no such component is possible. These phases are those in which either the driver and connector or the follower and the connector are in the same straight line. They are known as *Dead Points* of the cycle.

When the driver and the connector are in the same line, the direction of motion of their common centro, which is always at right angles to the line of the driver, is also at right angles to the line of link of the connector. A vector so placed with respect to a line can have no component along that line. (See Resolution of Motion, Chapter II.) Therefore at such a phase the driver has no influence over the follower and hence a dead point is reached.

An example of this condition is shown by that phase of Fig. 54, in which the driver is located at A_1. Other examples are given in phases *1* and *7* of Fig. 59 where the crank is located with the connecting rod along the center line of the engine. When such a dead point occurs in the steam engine mechanism, the engine is said to be on *Dead Center*. With such dead points as these, the inertia of the moving parts carries the mechanism beyond the dead-point phase and immediately thereafter the driver can compel motion of the follower.

Another condition arises when the lines of links of the follower and connector are in the same straight line: any component of motion directed along the length of the connector would tend to cause the follower to move either away from its center or toward it. Not a single point of the follower can be thus moved, for the follower is a rigid body and its center is a fixed center. Hence at such a dead-point phase the mechanism can have no motion. Examples of this type of dead point are shown in Fig. 54 at those phases in which the crank is at A_2 and A_3.

Because of the dead-point phases in the linkage of the type shown in Fig. 54, the motion of the follower C is restricted in the practical application of this linkage, so that the pin at bc travels a somewhat shorter distance than the theoretical arc length m_1m_3.

Compound Linkwork. In general, any linkage which is made up of more than four links is called a *Compound Linkage* or *Linkwork.* It is formed in reality by the addition of one or more links to some type of an elementary four-bar linkage. (See Figs. 34, 55, and 56.)

A practical application of a compound linkage in machine construction is shown in Fig. 55. This is an illustration of a deep-well pump manufactured by the Deming Company of Salem, Ohio. It employs a six-link mechanism, the skeleton diagram of which is shown in Fig. 56. Referring to both Figs. 55 and 56 it will be seen that link *A* is the crank of the crank shaft of the machine. It is the driver of the six-link mechanism and derives its motion through a series of gears (a train of gears) from a motor. Link *A* rotates (with respect to the frame *F*) about the fixed center *af*, which is the projection of the axis of the crank shaft or its bearings in the frame. The connector *B* connects *A* to the overhead beam or rocker, *C*, at the permanent center *bc*. Link *C* oscillates, with respect to the frame, about the fixed center *cf*. The slider *E* of the layout represents both the cross-head and the plunger of the pump and is constrained by the frame F, as shown, so that it has a rectilinear reciprocation along a vertical line. The slider derives its motion from *C* through the connector *D*.

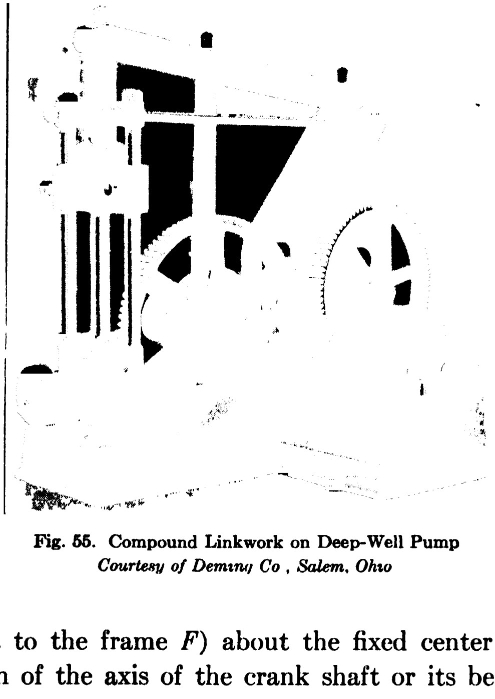


Fig. 55. Compound Linkwork on Deep-Well Pump
Courtesy of Deming Co., Salem, Ohio

In analyzing this mechanism it will be noted that links *A*, *B*, *C*, and *F* comprise a four-bar linkage in which *A* is the driver; *B*, the connector; *C*, the follower; and *F*, the frame. This elementary linkage of our six-link mechanism belongs to type (*c*) of the preceding article, in that rotation of the driver *A* produces oscillation of the follower *C*.

It will be noted also that links *C*, *D*, *E*, and *F* comprise another elementary linkage in which *C* is the driver; *D*, the connector; *E*, the follower; and *F*, the frame. Of course *F* must be the frame because a machine can have but one fixed link, which is necessarily

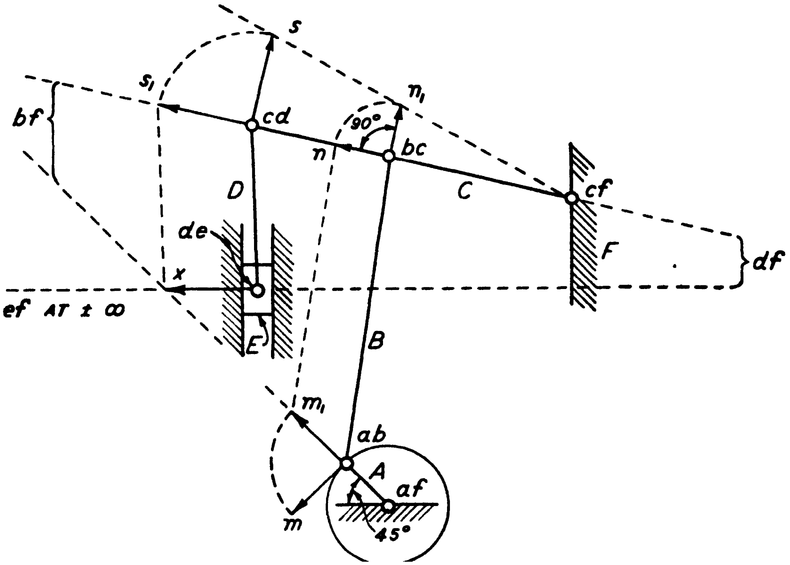


Fig. 56 Layout of the Six-Link Mechanism of Fig 55.

the fixed link of every mechanism of the machine. This second elementary mechanism of our six-link mechanism is a form of type (*d*) of the preceding article. However, in this case, oscillation of the driver *C* is changed into rectilinear reciprocation instead of rotative reciprocation (oscillation).

Thus it is seen that two four-bar linkages have combined to form a six-bar linkage by having one link thereof (which is *C* in this case) act as both driver and follower.

Since elementary linkages or mechanisms behave the same whether acting alone or in a combination with others, the study of a compound linkage or mechanism resolves itself into a study of the elementary forms of which it is composed. Hence a thorough understanding of elementary mechanisms is of fundamental importance.

Relative Velocities in a Compound Linkwork. An example of such velocities is the six-link mechanism of Fig. 56, in which the angular velocity of the driver *A* is assumed by the (linear) velocity of its pin at *ab*. Thus the vector *ab-m*, representing this velocity, is assumed at the 45-degree phase of the crank as shown in the figure. Solve this graphically for the velocity of the follower, the crosshead *E*.

Analysis. Since the velocities of all points of the slider *E* are the same, *de* is chosen as its representative point, for *de* is not only a point of *E*, but also of *D*. The links lying between link *A* (the source of motion) and link *D*, are *B* and *C*; and the common centros involved with these links are *ab*, *bc*, and *cd*. These common centros are noted because they make possible a comparison of velocities in order from *A* to *D* as follows:

From the known velocity of *ab*, the velocity of *bc* is obtained by considering *ab* and *bc* as points of the link *B*, which link rotates (relative to the frame) about the instant center *bf*.

From the velocity of *bc*, next is obtained the velocity of *cd* by considering *bc* and *cd* as points of the link *C*, which link rotates (relative to the frame) about *cf*.

From the velocity of *cd* the required velocity of *de* is now obtained by considering *cd* and *de* as points of the link *D* which rotates relative to the frame about *df*.

It should be noted that such a "link-to-link" method as this is applicable to any linkage, whether elementary, like the four-bar linkage, or compound.

Solution. Step 1. Preliminary work.

Lay out the given phase of the mechanism and locate, thereon, those instant centros which are necessarily used in the solution and in the proof thereof. These are the instant centers *af*, *ab*, *bc*, *cf*, *cd*, *de*, and *ef*, which can be obtained by inspection, and the instant centers *bf* and *df*, which can be obtained by the use of Kennedy's theorem. Draw *ab-m* perpendicular to the radius *af-ab* at *ab*.

Step 2. To find the velocity of *bc:*

Revolve *ab-m* into the radius *bf-ab*, *m* taking the position m_1.

Through m_1 draw line m_1n, parallel to the line of the link *B*, intersecting radius *bf-bc* at *n*. Then *bc-n* represents the velocity of the common centro *bc*. Why?

Step 3. To find the velocity of *cd*:

Revolve *bc-n* until it takes the position *bc*-n_1, at right angles to the radius *cf-bc*. Draw line *cf*-n_1 and extend it until it intersects (at *s*) a line *cd-s*, which is perpendicular to the radius *cf-cd*. Then *cd-s* represents the velocity of *cd*. Why?

Step 4. To obtain the velocity of *de*, which is the required velocity of the slider:

Revolve *cd-s* into the radius *df-cd*, *s* taking the new position s_1. Through s_1 draw line s_1x, parallel to *cd-de* and intersecting the radius *df-de* (extended) at *x*. Then *de-x* represents the velocity of the slider. *Ans.*

Note. Generally, *de-x* is left in this position (not revolved into its true position) so that the velocities of the slider, when found for several phases, will all lie parallel to each other. If each were revolved into its true position, they evidently would coincide; hence one would cover up another.

What is the direction of motion of the slider at the phase shown in the figure?

QUESTIONS AND PROBLEMS

1. State the meaning of formula (**12**).

2. What geometric theorem relative to similar triangles allows their use in the graphical solution of problems involving formula (**12**)?

3. In the example of Fig. 41, the vectors *Am* and *Bn* are 1½ inches and $^{15}/_{32}$ inch in length respectively. If a velocity scale of 1 inch = 600 feet per minute were used, what would be the velocities of points *A* and *B*? *Ans.* 900 f.p.m.; 281¼ f.p.m.

4. A disk like that in Fig. 42 is 4 inches in diameter and rotates about its geometric center. It carries a pin whose center, *A*, is located 1½ inches from the center of the disk. Point *B* of the disk is at a distance of ⅜ inch from its center. The velocity of the pin is represented by vector *Am*, whose length is 1¼ inches. Solve graphically for the length of the velocity vector of point *B*. *Ans.* $^5/_{16}$ in.

5. If the velocity of the pin in the preceding problem is 400 feet per minute, find

(*a*) the velocity scale used.
(*b*) the velocity of point *B* in feet per minute.
(*c*) the angular velocity of the disk in radians per minute. Use formula (**17**).
(*d*) the r.p.m. of the disk. Use formulas (**11**) and (**14**)

Ans. (*a*) 1 in. = 320 f.p.m. (*b*) 100 f.p.m. (*c*) 3200 radians per min. (*d*) 509.3 r.p.m.

Note. In the following graphical problems dealing with Fig. 57, it will be noted that the phase to be used is given or determined by the position of the line of link of the driver *A*. The latter is located by the angle between itself and the line of the frame. The method of laying out these phases is as follows:

Locate the fixed centers of driver and follower at a distance apart equal to the length of the line of the frame *D*.

Draw the driver in its given position with its length as stated below the figure.

With the fixed center of *C* as a center and a radius equal to the length of *C*, describe an arc which is the path of *bc*.

With *ab* as a center and a radius equal to the length of the connector *B*, describe an arc intersecting the path of *bc*. This point of intersection may then be connected to points *ab* and *cd*, and the lines thus drawn will be the positions of *B* and *C* respectively at the given phase. Be sure to locate and name all the instant centers that are necessary in effecting the solution or proving the correctness thereof.

When the resulting vector is found in each of these problems, place it in its correct position in relationship to its radius.

Scale: full size should be adopted as the scale of the drawing in all graphical problems (unless otherwise noted).

6. Assume a vector 1 inch in length to represent the velocity of centro *ab* of Fig. 57(*a*) and solve for the velocity (vector) of point *bc*.

7. Assume a vector 1 inch in length to represent the velocity of point *K* of the driver *A* of Fig. 57(*b*) and solve for the velocities of points *M* of link *B*, and *N* of link *C*.

8. Assume a vector ¾ inch in length to represent the velocity of *ab* of Fig. 57(*c*) and solve for the velocity of point *bc*. *Ans.* 1.02 in.

9. Assume a vector 1 inch in length to represent the velocity of *ab* of Fig. 57(*d*) and solve for the velocity of *K* which is a point on the line of the follower *C*.

Note. Obtain the velocity of *bc* first and from it determine the velocity of *K*.

10. Assume a vector ¾ inch in length to represent the velocity of *ab* of Fig. 57 (*e*) and solve for the velocity of *bc*.

11. Fig. 57(*f*) is another phase of the same linkage used in Fig. 57(*e*). It will be noted that the line of the driver is coincident with the line of the frame. As in problem 10, assume a vector ¾ inch in length to represent the velocity of *ab* and solve for the velocity of *bc*.

12. Locate the instant center *ac*, in both Figs. 57(*e*) and 57(*f*).

13. If, in Problem 10, the length of the vector representing the velocity of *bc* is found to be 1.02 inches, and driver *A* is making 120 revolutions per minute, find

(*a*) the linear velocity of *ab* in feet per minute. Use formula (**11**).

(*b*) the velocity scale when the vector which represents the (linear) velocity of *ab* is ¾ inch in length.

(*c*) the numerical value of the (linear) velocity of *bc*. Obtain this by multiplying the length of the vector of *bc*, which is given as 1.02 inches, by the scale found in part (*b*).

(*d*) the angular velocity of the driver *A* at the given phase in radians per second. Use formula (**17**).

(*e*) the angular velocity of the follower *C* at the given phase in radians per second. Use formula (**17**).

(*f*) the angular velocity ratio of driver *A* to follower *C* at the given phase.

Ans. (*a*) 47.1+ f.p.m. (*b*) 1 in. =62.8 f.p.m. (*c*) 64.1 −f.p.m. (*d*) 12.6− radians per sec. (*e*) 9.8− radians per sec. (*f*) $\frac{12\ 6}{9\ 8}$ or 1.3−

14. Assume a vector 1 inch in length to represent the velocity of the crank pin *ab* of Fig. 58(*a*), and solve for the velocity (vector) of the slider. Locate all instant centers.

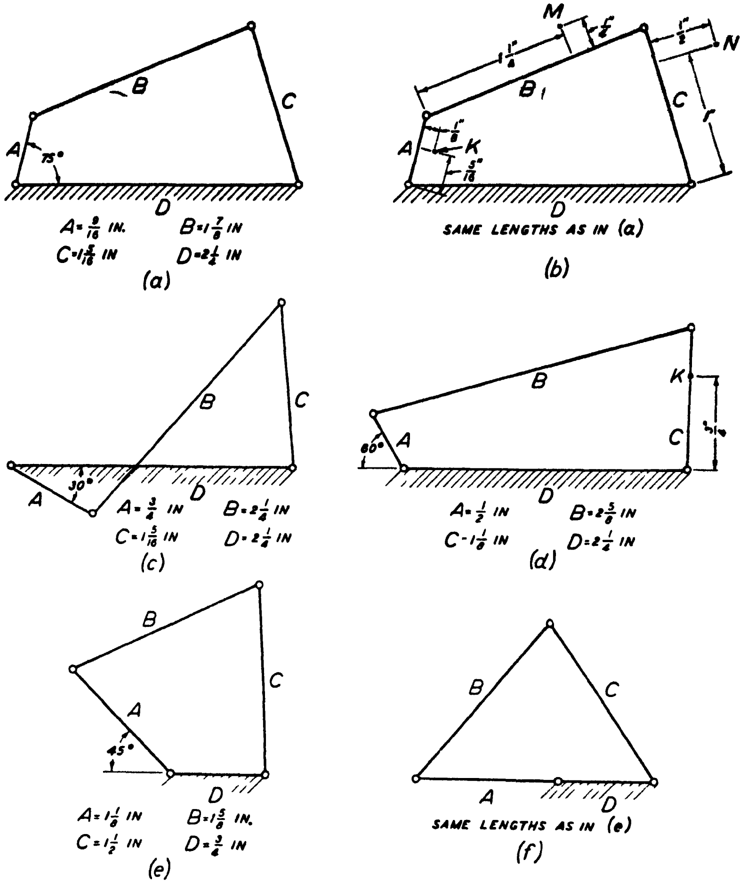

A
B
C
D
75°
A = 9/16 IN.
B = 1 7/8 IN
C = 1 5/16 IN
D = 2 1/4 IN
(a)
M
N
1 1/4"
1/4"
1/2"
1/8"
5/16"
K
A
B1
C
r
D
SAME LENGTHS AS IN (a)
(b)
A
B
C
D
30°
A = 3/4 IN
B = 2 1/4 IN
C = 1 5/16 IN
D = 2 1/4 IN
(c)
A
B
C
D
K
60°
3/4"
A = 1/2 IN
B = 2 5/8 IN
C = 1 1/8 IN
D = 2 1/4 IN
(d)
A
B
C
D
45°
A = 1 1/8 IN
B = 1 5/8 IN.
C = 1 1/2 IN
D = 3/4 IN
(e)
A
B
C
D
SAME LENGTHS AS IN (e)
(f)

Fig 57

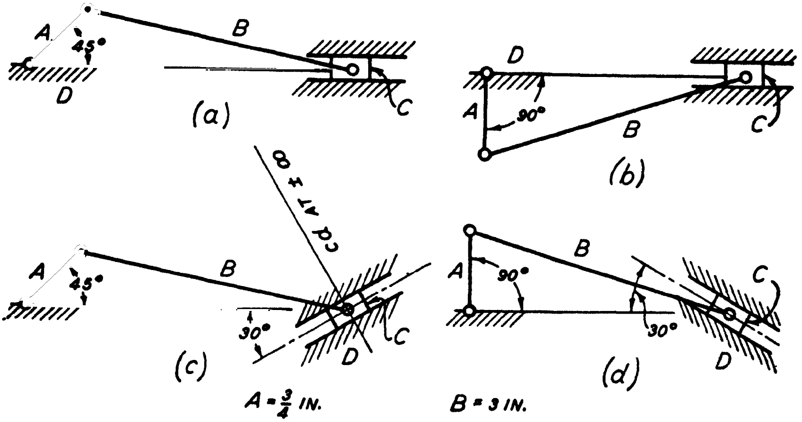

A
B
C
D
45°
(a)
A
B
C
D
90°
(b)
cd AT ∞
A
B
C
D
45°
30°
(c)
A
B
C
D
90°
30°
(d)
A = 3/4 IN.
B = 3 IN.

Fig 58

15. Assume a vector 1 inch in length to represent the velocity of the crank pin *ab* of Fig. 58(*b*) and solve for the velocity of the slider. Locate all instant centers.

16. Let the velocity of the crank pin of Fig. 58(*a*) be represented by the length of the crank, and solve for the velocity of the slider. Use the method of Fig. 49.

17. Let the velocity of the crank pin of Fig. 58(*b*) be represented by the length of the crank, and solve for the velocity of the slider. Use the method of Fig. 49.

18. What is the ratio of the velocity of crank pin to the velocity of slider in the phase shown in Fig. 58(*b*)? At what other phase will this same ratio occur?

19. Assume a vector $\frac{3}{4}$ inch in length to represent the velocity of *ab* in Fig. 58(*c*) and solve for the velocity of the slider. Locate all instant centers.

20. Assume a vector $\frac{3}{4}$ inch in length to represent the velocity of *ab* in Fig. 58(*d*) and solve for the velocity of the slider. Locate all instant centers.

21. Locate a point, *M*, on the line of the link of the connector of Fig. 58(*a*) midway between the end points of the link. Assume a vector 1 inch in length to represent the velocity of *ab* and solve for the velocity of point *M*.

22. Assume that the vector representing the velocity of the piston of a steam-engine mechanism in a graphical solution turns out to be $\frac{27}{32}$ inch in length when the vector of the crank-pin velocity is assumed as 1 inch. If the crank is 8 inches in length and makes 180 revolutions per minute, find

(*a*) the (linear) velocity of the crank pin in feet per minute. Use formula (11).

(*b*) the angular velocity of the crank pin in radians per second. Use either formula (14) or formula (17).

(*c*) the angular velocity of the crank in radians per second.

(*d*) the velocity scale used in the graphical solution.

(*e*) the linear velocity of the piston in feet per minute.

Ans. (*a*) 754 − f.p.m. (*b*) 18.8 radians per sec. (*c*) 18.8 radians per sec. (*d*) 1 in. = 754 f.p.m. (*e*) 636.2 f.p.m.

23. Make a statement of the Angular Velocity Theorem as applied to the four-bar linkage.

24. State the conditions for a constant angular velocity ratio between driver and follower in a four-bar linkage.

25. What is a *compound linkwork?*

26. Reproduce the layout of the six-link mechanism of Fig. 56, enlarging it to twice the size given by the dimensions of Problem 17 of Chapter III. Assume a vector *ab-m*, $\frac{3}{4}$ inch in length, to represent the velocity of *ab* and solve graphically for *de-x* which represents the velocity of the slider *E*.

27. Same as Problem 26, but use the phase of the mechanism as given in Problem 18 of Chapter III.

28. Same as Problem 26, but use the phase of the mechanism as given in Problem 19 of Chapter III.

CHAPTER V

VELOCITY DIAGRAMS

Introduction. In the preceding chapter it has been seen that the instantaneous velocity of a point (that is, its velocity at a given phase) can be determined graphically. The method used in determining such a velocity can be repeated at other phases distributed throughout the cycle of the mechanism, and the velocities thus established can be so coordinated that a graph can be drawn. This graph is called a Velocity Diagram. It furnishes at a glance an idea of the manner in which the velocity of the point may be changing, and provides definite values of velocities, not only at the successive phases which were considered or used in its construction, but also at intervening phases as well. It is therefore a very important device or instrumentality for studying and recording relative velocities.

In order that the successive velocities of a point be brought together in a velocity diagram, the driver of the mechanism must have, or be assumed to have, a constant angular velocity. Since this angular velocity is represented graphically by the corresponding linear velocity of some point of the driver, the linear velocity in turn must be constant for all phases considered. Hence the point of the driver, whose velocity is known or assumed, must be represented at every phase throughout the cycle by the same length of vector.

Since in any velocity diagram it is necessary to obtain instantaneous velocities at successive phases throughout the cycle, a number of these phases are first selected by placing the driver in positions which are equally spaced about its fixed center. Equally spaced positions of the driver do not necessarily provide a good distribution of positions or phases of the follower or point whose velocity diagram is required. Other phases can be introduced later on if it is found that they are necessary for the determination of the diagram. Having laid out the positions of the driver and having numbered them, it is well to draw only one complete general phase of the mechanism. An analysis of the problem can be made for this phase, and a mechanical method of effecting the solution can be devised. The phase is then solved. Its result should be numbered by the number of the phase that is given to the position of the driver. The other phases may now be solved in the order of the positions of their drivers. The analysis of the problem is the same for all phases, and the mechanical method of effecting the solution in one phase may be used in the others as a rule. However, occasionally a special case arises which demands a slightly different mechanical method.

Velocity diagrams can be classified under three types, as follows:

Velocity-space diagrams
Polar-velocity diagrams
Radial velocity diagrams

Velocity-Space Diagram of Slider-Crank Mechanism. The case of slider-crank mechanism to be considered first is one in which the path of the slider is along a line that passes through the fixed center of the crank. As noted previously in this text, such a slider-crank mechanism is known as the steam-engine mechanism. The problem is to obtain the velocity-space diagram of the cross-head (or piston) which is the slider of the mechanism.

Referring to Fig. 59, crank positions are shown laid out and numbered from *1* to *12* inclusive. Having selected phase *3* as a good representative phase or general case (phases *1* and *7* are not quite so general, for the crank and connector are in the same straight line), it is laid out on the drawing for use in analyzing the problem and adopting the mechanical procedure for the graphical method of solution. This has been treated fully in Chapter IV (see Fig. 48) and so will not be repeated here. It should be noted by the student that phase *3* is shown in full on Fig. 59 and that it is similar to that of Fig. 48.

Having obtained the velocity of *bc* (the slider) at phase *3* in which it is represented by $bc\text{-}n_3$, it is marked by the number $3'$ so as to connect it with its phase. The rest of the phases are then laid out in order and each one of them is solved in exactly the same manner as phase *3*. In so doing, a circle is described with *ad* as a center and $ad\text{-}m_3$ as a radius, so that at each of the other phases the extension of the crank to this circle gives the revolved position of *ab-m* for that phase. It is possible to do this, for it must be re-

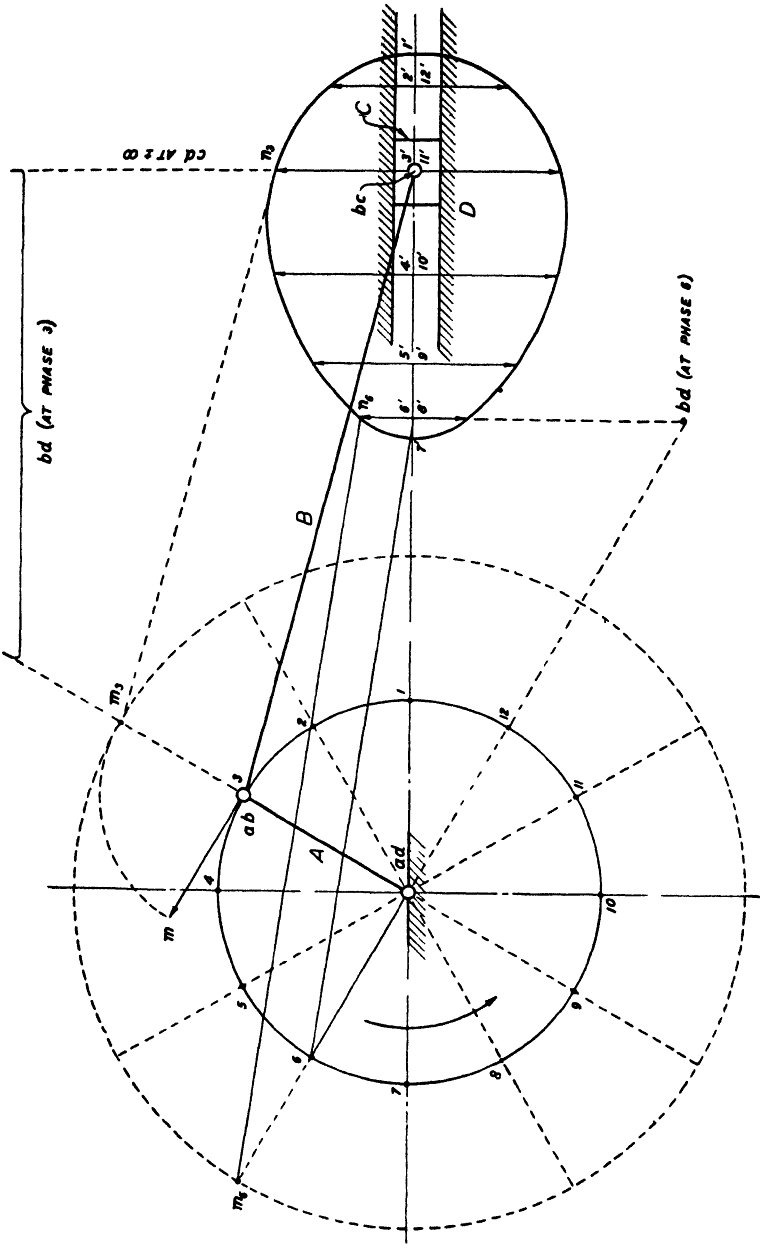


Fig. 59 Velocity-Space Diagram

membered that *ab-m* is constant, since the driver is assumed to have a constant angular velocity. It should be noted that, in phase *3*, *ab-m* is rotated clockwise into the radius *bd-ab*. It could be rotated counterclockwise, but the direction of rotation which is used in one phase must be used throughout all other phases of the same problem. The use of the circle described with $ad\text{-}m_3$ as a radius and with *ad* as a center, not only facilitates the work in general, but also secures the same direction of rotation of *ab-m* at all phases.

Phase *6* of the problem is shown on the figure by light solid lines. The location of the connector *B* is obtained as follows: with the outer end of the crank as a center and the length of the connector *B* as a radius, describe an arc intersecting the path of *bc* to give position *6′* of the slider. Line *6-6′* is then the new position of *B*. Just as m_3n_3 was drawn parallel to the position of *B* at phase *3*, so now m_6n_6 is drawn parallel to position *6-6′* of the connector *B*, intersecting the line from *6′* to *bd* (at phase *6*) at the point n_6. Thus $6'n_6$ represents the velocity of the slider at phase *6*. (Be sure to note that m_6n_6 is not parallel to the position of *B* of phase *3* and hence is not parallel to m_3n_3.)

Two important phases of any mechanism using a slider are those phases at which the slider is at the end of its stroke and hence has a zero velocity. These phases are numbers *1* and *7* of Fig. 59. Such phases must always be obtained. While it is evident that the velocity of the slider is zero at these phases, it is interesting to note that the general method of solution can be used therein.

Having solved each phase of the cycle, and having obtained the resulting velocities of the slider in the positions as shown, the end points of these are connected and the graph thus obtained is called a *Velocity-Space Diagram.* Enough points must be had to thoroughly define the graph in every case. If it is seen that the original layout of phases is insufficient to completely determine the graph, other phases can be inserted where it is evident that they are needed.

It is evident from the above that a velocity-space diagram is one in which the lines representing the velocities are so set up or obtained that they take positions perpendicular to the path of the point whose velocities they represent.

A velocity-space diagram of another slider-crank mechanism is given in Fig. 60. The links are named as in the preceding examples

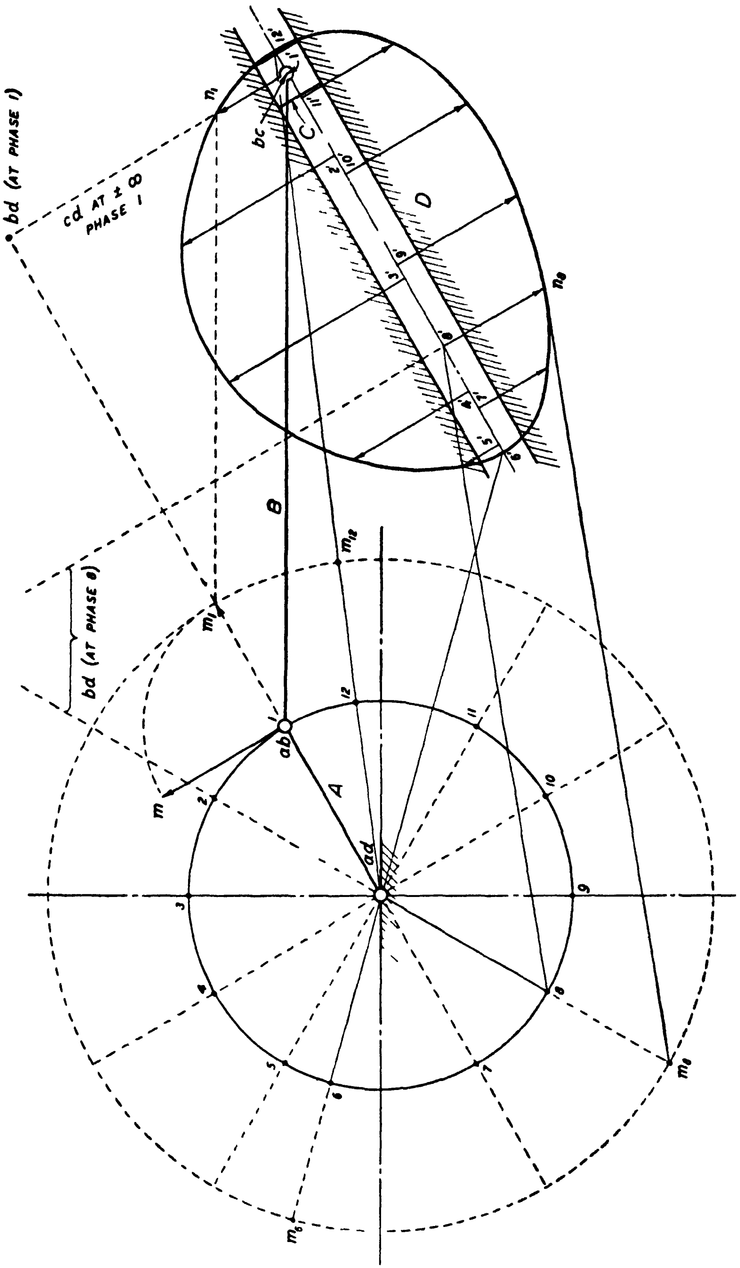


Fig. 60. Velocity-Space Diagram

of slider-crank mechanisms. In this case the slider, *C*, reciprocates along a path that is at an angle of 30° with the horizontal. The constant angular velocity of the crank or driver *A* is given by *ab-m*, the constant linear velocity of the crank pin at *ab*. Obtain the velocity diagram of the slider *C* (or its centro, *bc*).

Having located *ad*, the fixed center of the crank, and having laid out the path along which the slider is to travel, the positions of the crank are selected for the several phases to be considered. As in Fig. 59, these phases are equally spaced with the exception of numbers *6* and *12*, which are the positions of the crank that dictate the ends of the stroke of the slider and hence the phases in each of which the velocity of the slider is zero. The ends of the stroke are therefore points on the velocity diagram.

To locate phase *6* so as to obtain the left end of the stroke of the slider, the length of the crank is subtracted from the length of the connector. With this difference as a radius, and the fixed center *ad* as a center, an arc is described which intersects the path of *bc* at its end point, *6'*. A line drawn from this point through *ad* to the crank-pin circle at *6* is the line of the connector. Therefore *ad-6* is the corresponding position of the crank.

To locate phase *12*, which will give the right end of the stroke, *12'*, the length of the crank is added to that of the connector. With this sum as a radius and *ad* as a center, an arc is described which intersects or cuts the path of the line *bc* at *12'*. A line drawn from *12'* to *12* is the line of the connector which corresponds to the position *ad-12* of the crank.

A representative phase such as number *1* of Fig. 60 is now selected so that a graphical method can be devised that may be followed through the successive phases in order to obtain the velocities of the slider. The method of procedure is exactly the same as in the previous problem of Fig. 60. The student should note however that since in this problem the direction of motion of the slider is along a line which makes an angle of 30 degrees with the horizontal, the radius *bd-bc* will not be a vertical line at any phase as was the case in Fig. 59. Fig. 60 shows every step in the solution for phases *1* and *8*.

Polar-Velocity Diagram of Steam-Engine Mechanism. The diagram of the steam-engine mechanism shown in Fig. 61 makes use of the velocity problem of Fig. 49. It shows the velocities of the piston or slider throughout the cycle of the mechanism.

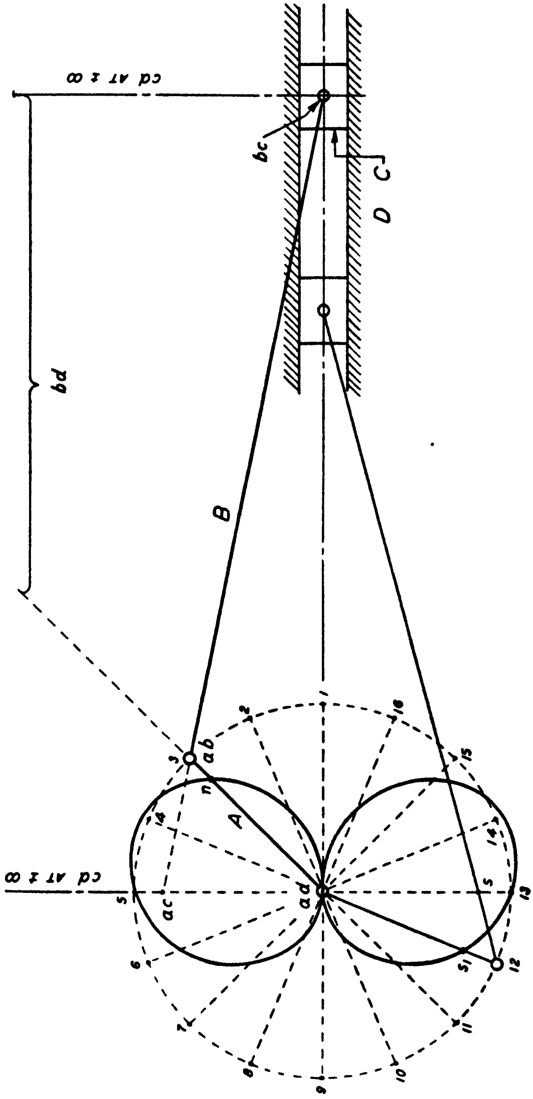


Fig 61 Polar-Velocity Diagram

A number of positions of the driver of the mechanism are located about the fixed center *ad*. A phase of the mechanism is set up for each position of the driver; phases *3* and *12* are given in full.

The constant velocity of the crank pin is, and must be, represented by the length of the crank. Then such segments of the vertical center line of the crank-pin circle as segments *ad-ac* of phase *3* and *ad-s* of phase *12* (point *s* of phase *12* is the centro *ac* at that phase) which lie between the fixed center *ad* and the point of intersection of the center line with the connector *B*, represent the velocity of the slider at each of these two phases respectively. Each segment is then laid off from *ad* on the position of the crank at that phase, so that *ac* takes the position *n*, and *s* takes the position s_1. Having done this for every phase, the points thus established on the crank positions are connected in order. The diagram formed is called a *Polar-Velocity Diagram* because the fixed center *ad* is used as a pole and the segments are in reality polar coordinates.

It will be noted that the polar-velocity diagram of this mechanism is made up of two equal lobes. Each of these lobes gives the velocities of the piston during one complete stroke or its equivalent (one-half of a revolution of the crank). It will be noted that each lobe extends beyond the crank-pin circle, which indicates that at certain phases the piston has a greater velocity than the crank pin.

Radial-Velocity Diagram of a Four-Bar Linkage. The four-bar linkage of Fig. 62 is made up of the driver *A*, the connector *B*, the follower *C*, and the frame *D*. The velocity diagram of the figure gives the velocities of the end *bc* (of the follower) throughout the cycle. These velocities of course are based on the constant angular velocity of the driver.

Representative phases are selected which should include those phases that give the limiting positions of the follower *C*, in case the follower has a motion of oscillation as in this problem. Each successive phase is then solved for the velocity of point *bc* by the method explained in Chapter IV. (See Figs. 44 and 46.) This method is shown completely by phase *2* of Fig. 62. The velocities are not revolved into their true positions, but are left as obtained, extending radially outward or inward from the respective positions of *bc*. The ends of the lines representing the velocities and the limiting positions of point *bc* (points *5′* and *10′* at which the velocity of *bc* is zero) are then connected by the curved line as shown. This curved line or graph is called a *Radial-Velocity Diagram.*

The radial-velocity diagram of a rotating follower of a link-connector mechanism is obtained by exactly the same method as that used in Fig. 62 for the oscillating follower. Since a rotating follower never reverses its path, it is evident that its radial-velocity diagram would lie entirely outside of, and hence surround, the circular path of the end of the follower.

A velocity diagram that is much more compact and serviceable than its radial-velocity diagram can be prepared for a rotating follower. Instead of leaving the velocities in the positions they occupy in the radial-velocity diagram, they are moved radially inward along their respective positions of the follower until they reach the fixed center of the follower. This fixed center then becomes a pole from which the velocities extend radially outward; hence by connecting their end points, a Polar-Velocity Diagram results. This is not feasible for an oscillating follower of a four-bar linkage.

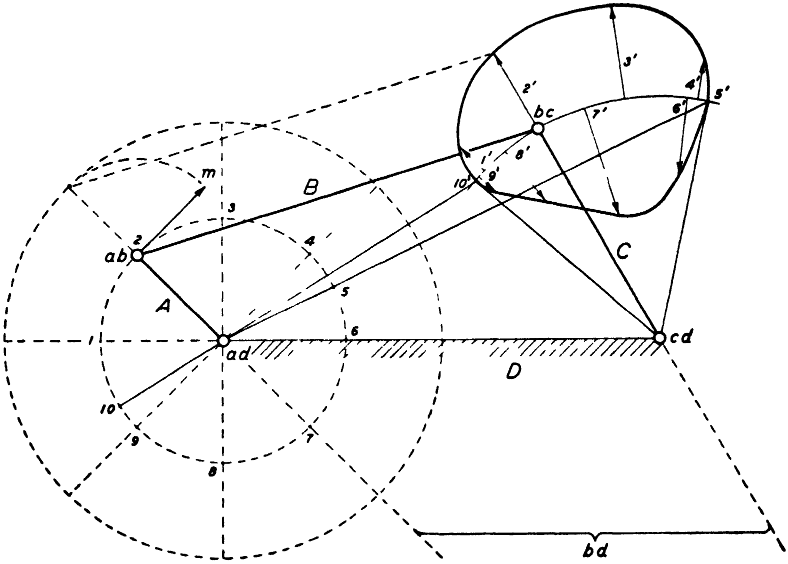


Fig. 62 Radial-Velocity Diagram

Velocity-Space Diagram of a Compound Linkage. In order to demonstrate the manner in which a velocity diagram of a compound linkage is obtained, a mechanism which is known as the Oscillating-Arm Quick-Return Motion has been selected. This is one of several forms of so-called quick-return mechanisms which are to be found in machine tools and various other kinds of machines. In machine tools like the shaper and slotter, a cutting tool moves back and

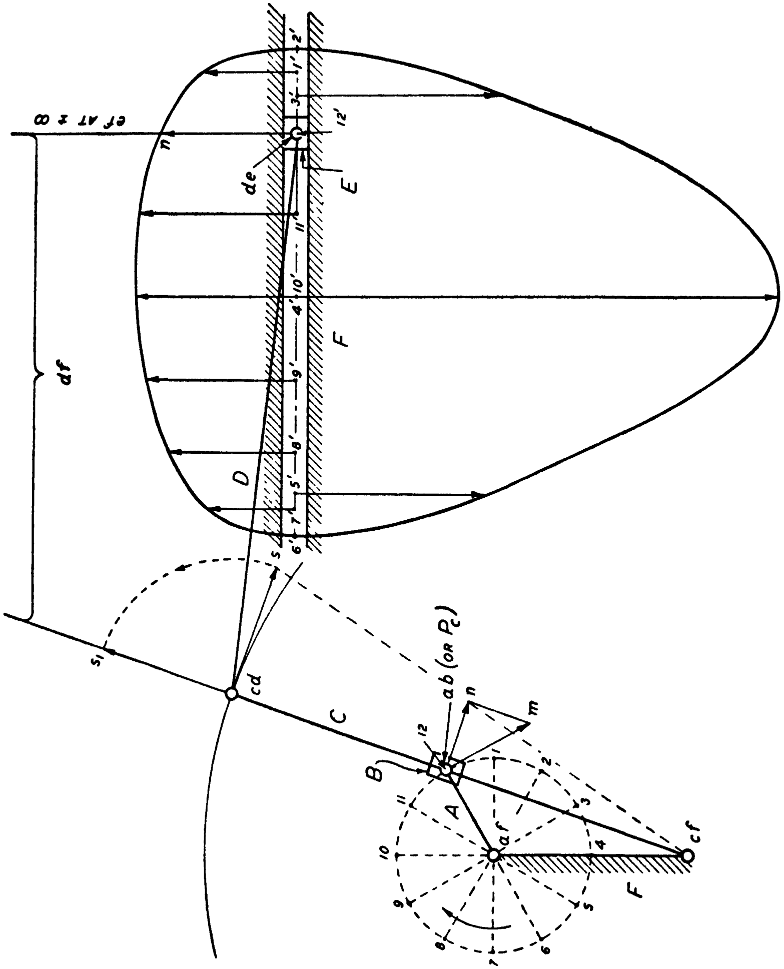


Fig. 63. Velocity-Space Diagram of a Compound Linkage

forth (reciprocates) taking its cut on one stroke (the working stroke) and returning on its second (idle) stroke so as to get into position for another cut. The mechanism used in such a machine should give to the cutting tool a speed as nearly uniform as possible on its working stroke in order to work well, and a higher speed on its return stroke so that little time is wasted.

This mechanism was described in conjunction with Fig. 34, and its instant centers were located. It is shown in Fig. 63 together with the velocity-space diagram of its slider (or tool) *E*.

In order to obtain this velocity diagram, proceed as follows:

Locate the fixed centers *af* of the driver *A*, and *cf* of the oscillating arm *C*. Next locate the path of the pin (centro *de*) of the slider *E*.

With *af* as a center and the length of the crank *A* as a radius, describe the crank-pin circle and locate thereon a number of crank positions which are to dictate the phases of the mechanism to be considered.

With *cf* as a center and the length of arm *C* as a radius, describe an indefinite arc along which the end *cd* of *C* will move.

Draw the positions of the oscillating arm (from *cf* of course) tangent to the crank-pin circle on either side and extending to the path of *cd*. These are the extreme positions of the oscillating arm *C*. Hence the limiting positions of the point *de* of *E* can be obtained by describing arcs with the ends of these tangent positions of *C* as centers and with a radius equal to the length of the connector *D*. These arcs intersect the path of *de* at points *2'* and *6'*, which are so named because it so happens that *C* falls tangent to the crank pin circle at the crank pin positions *2* and *6*. Since the velocity of the slider *E* must be equal to zero at points *2'* and *6'*, the latter are points on its velocity diagram.

Lay out a representative phase of the mechanism such as that which uses crank position *12*. Analyze it carefully so as to devise a graphical method (given later) by which the velocity of *E* can be obtained from the known or assumed constant velocity of the crank pin at *ab*. Having obtained the velocity of *E* at this phase, lay out and solve each of the other phases in order. The velocity diagram may then be drawn through the end points of those lines which represent the velocities of the slider *E*. If not enough points are at hand to properly define the diagram, this can be remedied by the selection and solution of more phases located where needed. Thus in the diagram of Fig. 63 there is too great a distance between the velocity at *3'* and that at *4'*, and between *4'* and *5'*. Hence one or more additional phases between crank positions *3* and *4*, and also between *4* and *5*, should be laid out and solved.

The solution of phase *12* will now be given and the centros shown in Fig. 63 belong to this phase.

Locate the phase by drawing first the link *C*, from *cf* through *ab* to *cd*. With *cd* as a center and the length of *D* as a radius, describe an arc intersecting the path of *de* at *12'*. This is the location of the pin of the slider *E* at phase *12*. Connect *cd* and *de* for the position of the connector *D* at this phase.

Lay out the vector *ab-m* representing the velocity of *ab* perpendicular to *af-ab* at *ab*.

As already stated, the arm *C* oscillates about the fixed center *cf*. Therefore the direction of motion of any point of *C* is along a line perpendicular to its radius drawn to *cf*. Since the sliding block *B* slides in a slot of *C*, some point of *C*, say P_c, is coincident with point *ab* of *B*. The velocity of P_c, which is along a perpendicular to its radius *cf*-P_c, must be the same as the velocity of *ab* along that perpendicular; for if it were not, *ab* would move off the line of the link *C*; this however is not possible because *B* cannot leave the slot of *C*. In other words, *B* is constrained by *C*. Since the velocity of *ab*, along a perpendicular to link *C*, must be a component of its given velocity, *ab-m*, the latter is resolved now into its two components, *ab-n* (or P_cn) and *nm*, respectively perpendicular and parallel to the line of the link *C*. (See Fig. 20 also.) Then P_cn represents the velocity of point P_c of link *C*.

Draw a line from *cf* through point *n* and extend it to *s* to cut off the perpendicular to *cf-cd* at point *cd*. Then *cd-s* represents the velocity of *cd*. Why?

Rotate *cd-s* about *cd* as center into its radius *df-cd*, where it takes the position *cd*-s_1.

Draw a line through s_1 parallel to the position of the connector *D* at the given phase. This line intersects the radius *df-de* at *n*, so that *de-n* represents the velocity of *de* (or *E*) at this phase to the same scale as that used for *ab-m*. Why?

Note. Since, in phase *12*, *cd-s* is rotated counter-clockwise, it must be rotated in that direction for all phases.

QUESTIONS AND PROBLEMS

1. What is the purpose of a velocity diagram?

2. In constructing a velocity diagram, is the motion of the driver assumed to be constant or variable?

3. Where are zero velocities registered in the velocity diagram of a sliding block?

4. Construct the velocity-space diagram for the piston of a steam-engine mechanism in which the lengths of crank and connecting rod are 1 inch and 4 inches respectively. Let the constant velocity of the crank pin be represented by a vector ¾ inch in length.

5. Draw the complete polar velocity diagram of the steam-engine mechanism of Problem 4. The velocity of the crank pin will be represented necessarily by the length of the crank.

6. Construct the velocity-space diagram of the slider of Fig. 58(*c*). Let the constant velocity of the crank pin be represented by a vector ⅝ inch in length.

7. Construct the velocity-space diagram of the slider of Fig. 58(*d*). Let the constant velocity of the crank pin be represented by a vector ⅝ inch in length.

8. In a four-bar linkage of the type shown in Fig. 62, the distance between the fixed centers is 3 inches, and the lengths of the driver *A*, connector *B*, and follower *C*, are ⅝ inch, 3½ inches, and 1½ inches respectively. Assume the constant velocity of the end *ab* of the driver to be represented by a vector whose length is ½ inch. Construct the radial-velocity diagram of the end *bc* of the follower.

9. Construct the velocity-space diagram of the slider *E*, of the oscillating-arm quick-return mechanism of Problem 12, Chapter III. Assume a vector ⅝ inch in length to represent the constant velocity of *ab*.

10. Construct the velocity-space diagram of the slider *E* of Problem 26, Chapter IV. (This deals with Fig. 56, and all dimensions as given in Problem 17, Chapter III, are to be multiplied by two.) Let the vector representing the velocity of *ab*, the end of the crank, be ¾ inch in length. See Problems 26, 27 and 28 of Chapter IV.

WRAPPING CONNECTORS

Courtesy of Morse Chain Co., Ithaca, New York

CHAPTER VI

WRAPPING CONNECTORS

Introduction. Any mechanism employing a flexible connector to communicate motion from the driver to the follower is called a *wrapping-connector* or *flexible-link connector mechanism.* Since the connector is a flexible band, it is only through a pull that motion of one member can be produced by the other. This type of mechanism is employed most frequently with parallel shaft installations.

Wrapping connectors may be classified as follows:

(*a*) Flat belts
(*b*) **V**-belts
(*c*) Ropes
(*d*) Chains

The links with which they are used in the transmission of motion are as a rule (for an exception see the general case of Fig. 64) circular in section. The rim of a link is shaped in a manner to accommodate the type of connector used with it; it may be flat or slightly crowned to take a flat belt, grooved to take a **V**-belt or rope, or toothed to take a chain. In general these links are spoken of as wheels, but may be distinguished from each other by such names as pulleys, grooved pulleys or sheaves, and sprockets.

The most general or basic case of a wrapping-connector mechanism is the one illustrated in Fig. 64, in which an oscillating motion of one member, or its shaft, is transmitted to the other member which is located on a parallel shaft. In this case the belt is not continuous and each wheel of the pair must be alternately driver and follower. While such a condition may occur in practice, it is seldom used; instead, a special case of the wrapping-connector mechanism is found in which the wheels are right circular cylinders, the flexible connector is an endless band, the wheels rotate about their geometric centers (or axes), and the angular velocity ratio between the wheels is theoretically constant. (See Figs. 66, 67, and 74.)

Positive Driving and Frictional Driving with Flexible Connectors. The transmission of motion in all wrapping-connector mechanisms is made possible by either: (*a*) positive driving or (*b*) frictional driving.

In the general case of Fig. 64, positive driving (*a*) is made possible by securing the band to the rim of each wheel. In the case of any chain drive, positive driving is secured through the action of the teeth of the sprockets with the links of the chain. Frictional driving (*b*) is secured by introducing an initial tension or pull into the belt or rope as it is placed (or stretched) over the pulleys, thereby setting up a frictional resistance between the belt and the rim of each pulley. With **V**-belts and textile ropes a wedge-like action is secured between the grooves of the sheave (pulley) and the belt or rope. This materially increases the frictional action and hence a smaller initial tension or pull is needed than that used for flat belts. Thus through either positive or frictional driving, the connector is supposed to have, as nearly as is possible from a practical standpoint, the same linear velocity as the rim would have at the mid-section of the connector.

The Pitch Diameter of a Pulley, Sheave, or Sprocket. The diameter of the wheel taken at the mid-section of the wrapping connector is sometimes called the *Pitch Diameter.* Hence the pitch diameter of a pulley of a flat-belt drive is equal to the diameter of the pulley plus the thickness of the belt. (See Fig. 66.) However, the pitch diameter is generally assumed as equal to the diameter of the pulley, since in flat-belt drives the thickness of the belt is so small it need not be considered. It is evident from Figs. 79, (*a*) and (*b*), 80, 84, and 85, that the pitch diameters of **V**-belt and rope sheaves depend on the manner in which the belt or rope rides in the groove. In the roller chain drive of Fig. 88 the pitch diameter is equal to the diameter of the sprocket at the bottom of the groove plus the diameter of the roller.

Angular Velocity Theorem. In investigating the angular velocity ratio of driver to follower in a wrapping-connector mechanism, the general case of the flat-belt drive which is illustrated in Figs. 64 and 65 will be considered. In these figures, belt *C*, considered inextensible, is wrapped only a part of the way around wheels *A* and *B*, one end of the belt being attached to each wheel. In this manner *A* and *B* act alternately as driver and follower with the result that both have a motion of oscillation about their fixed centers *ad* and *bd.*

Since flat belts are relatively thin, the belt in the skeleton diagram of Fig. 65 is represented by its inside edge, which is tangent to the wheels at the points of tangency *ac* and *bc.* These points are so named because they are, at the given phase, the instant centers of the links *C* and *A*, and *C* and *B* respectively. For that reason also, at the given phase, line *ad-ac* represents the link *A*, and line *bd-bc* represents the link *B* in the skeleton diagram. The other instant centers, *cd* and *ab*, are found by the application of Kennedy's theorem. It should be noted that the instant center *ab* must be on both the line of the connector, *ac-bc*, and the line of fixed centers, *ad-bd*. Therefore it must be at the intersection of these two lines. Hence it is the point at which the line of the connector *C* cuts the line of fixed centers externally into the segments *bd-ab* and *ad-ab*.

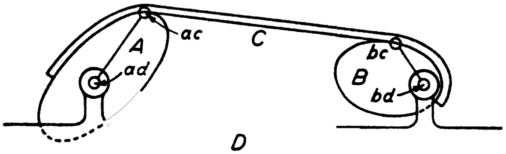


Fig 64

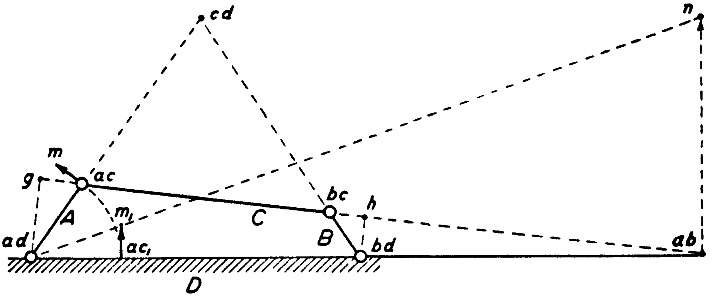


Fig 65

In Fig. 65 let *ac-m*, drawn perpendicular to *ad-ac* (the radius of *ac*) represent the linear velocity of *ac*. As previously explained, this velocity is actually equal to *ac-m* multiplied by whatever velocity scale is in use. However, for the sake of brevity, the vector representing a velocity will henceforth be considered and spoken of as the velocity itself.

Revolve *ad-ac*, together with the vector *ac-m*, about *ad* as a center, until *ad-ac* takes the position *ad*-ac_1 on the line of fixed centers, and *ac-m* takes the position ac_1-m_1, perpendicular of course to *ad*-ac_1. Draw line *ad*-m_1, extending it until it intersects, at *n*, the line *ab-n* which is drawn perpendicular to *ad-ab*.

In the similar right triangles $ad\text{-}ac_1\text{-}m_1$ and $ad\text{-}ab\text{-}n$,

$$\frac{ab\text{-}n}{ac_1\text{-}m_1}=\frac{ad\text{-}ab}{ad\text{-}ac_1}$$

In this proportion

$ac_1\text{-}m_1$ = the linear velocity of ac

$ad\text{-}ac_1$ = the radius of ac as a point of link A

$ad\text{-}ab$ = the radius of ab as a point of link A.

Therefore from formula (**12**),

$ab\text{-}n$ = the linear velocity of ab

Letting V_{aa} represent the angular velocity of link A, and V_{ab} represent the angular velocity of link B, we have from formula (**17**)

$$V_{aa}=\frac{ab\text{-}n}{\text{the radius of } ab \text{ as a point of } A}$$
$$=\frac{ab\text{-}n}{ad\text{-}ab}$$

and

$$V_{ab}=\frac{ab\text{-}n}{\text{the radius of } ab \text{ as a point of } B}$$
$$=\frac{ab\text{-}n}{bd\text{-}ab}$$

so that

$$\frac{V_{aa}}{V_{ab}}=\frac{\dfrac{ab\text{-}n}{ad\text{-}ab}}{\dfrac{ab\text{-}n}{bd\text{-}ab}}$$
$$=\frac{ab\text{-}n}{ad\text{-}ab}\times\frac{bd\text{-}ab}{ab\text{-}n}$$
$$=\frac{bd\text{-}ab}{ad\text{-}ab} \qquad (\mathbf{23})$$

Since these radii, **bd-ab** and **ad-ab**, are the segments into which the line of fixed centers of the mechanism is cut by the connector C, formula (**23**) may be stated as follows:

In any wrapping-connector mechanism, at any phase, the angular velocities of driver and follower are to each other inversely as the segments into which the line of fixed centers is cut by the line of the connector.

In Fig. 65, lines **ad-g** and **bd-h** are drawn from the fixed centers **ad** and **bd**, respectively, perpendicular to the line of the connector C.

Two similar right triangles, *ad-g-ab* and *bd-h-ab*, are thus formed. In these triangles,

$$\frac{bd\text{-}ab}{ad\text{-}ab}=\frac{bd\text{-}h}{ad\text{-}g}$$

Therefore from formula (23)

$$\frac{V_{aa}}{V_{ab}}=\frac{bd\text{-}h}{ad\text{-}g} \tag{24}$$

Formula (24) states that in any wrapping-connector mechanism, at any phase, the angular velocities of driver and follower are to each other inversely as the perpendiculars which are drawn from their fixed centers to the line of the connector.

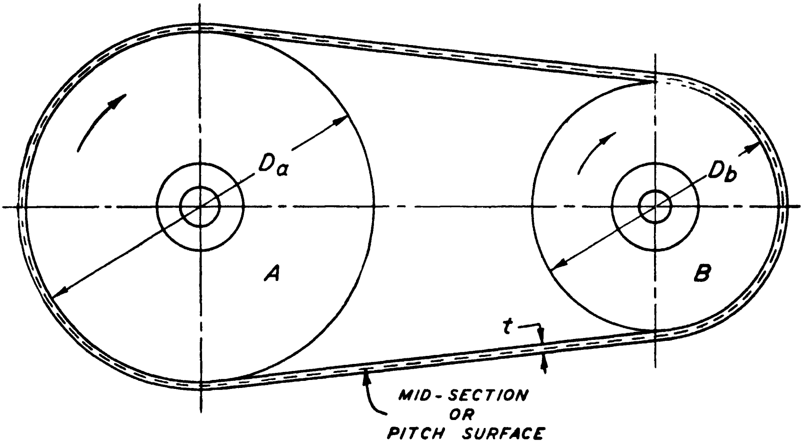


Fig 66 Open Flat-Belt Drive

The statements of formulas (23) and (24) combine to form the statement of the Angular Velocity Theorem for a wrapping-connector mechanism.

Constant Angular Velocity Ratio. It is evident from formula (23) that if $\frac{V_{aa}}{V_{ab}}$ (the angular velocity ratio of driver to follower) is constant, $\frac{bd\text{-}ab}{ad\text{-}ab}$ (the ratio of the segments of the line of fixed centers) must likewise be constant. Since *ad* and *bd* (see Fig. 65) are fixed points on the line of centers, the ratio of these segments can be constant when, and only when, the segments themselves do not vary in length from phase to phase throughout a cycle. This will be the case if the line of the connector always intersects the line of fixed centers at the same point to give the common centro, *ab*, a fixed position on the line of centers.

Therefore the condition for a Constant Angular Velocity Ratio between driver and follower in a wrapping-connector mechanism is that their common centro must be at a fixed position on the line of centers throughout the cycle.

This condition is met with when the driver and follower are circular in section and rotate about their geometric centers, as is the case in both the open-belt drive of Figs. 66 and 68 and the crossed-belt drive of Fig. 69. It will be noted in Figs. 68 and 69 that the connector tT cuts the line of fixed centers OO_1, at the same point at every phase. Of course for continuous rotation of driver and follower an endless belt must be used, in which case the action is frictional and not positive as in the attached belt of Fig. 64.

Angular Velocity Ratio in Open-Belt and Crossed-Belt Drives. In Figs. 68 and 69 the belt is shown tangent to pulleys A and B at T and t respectively, and O_1T and Ot are radial lines drawn to these points of tangency. Since a tangent to a circle is perpendicular to the radius drawn to the point of tangency, O_1T and Ot are the perpendiculars dropped from the fixed centers O_1 of pulley A, and O of pulley B, respectively, to the line of the connector, tT. Hence from the statement of formula (24), with V_{aa} representing the angular velocity of pulley A and V_{ab} representing the angular velocity of pulley B,

$$\frac{V_{aa}}{V_{ab}}=\frac{Ot}{O_1T} \tag{a}$$

Since at every phase O_1T is equal to the radius R_a of A, and Ot is equal to the radius R_b of B,

$$\frac{O_1T}{Ot}=\frac{R_a}{R_b}=\text{a constant} \tag{b}$$

From steps (*a*) and (*b*)

$$\frac{V_{aa}}{V_{ab}}=\frac{R_b}{R_a}=\text{a constant} \tag{25}$$

Let $D_a=2R_a=$ the diameter of pulley A, and $D_b=2R_b=$ the diameter of pulley B. Multiplying both numerator and denominator of the second member of formula (25) by 2,

$$\frac{V_{aa}}{V_{ab}}=\frac{2R_b}{2R_a}=\frac{D_b}{D_a}=\text{a constant} \tag{26}$$

From formulas (25) and (26) it may be stated that, in any belt drive, the angular velocity ratio of driver to follower is constant and is inversely as the radii and inversely as the diameters of pulleys.

R.p.m. Ratio in Open-Belt and Crossed-Belt Drives. In a belt drive, the driving pulley imparts its rim speed (velocity) to the belt and the latter in turn imparts its speed to the rim of the driven pulley. Hence the rim velocities of the pulleys are equal to each other and to the belt speed as well. Referring to Figs. 68 and 69, let

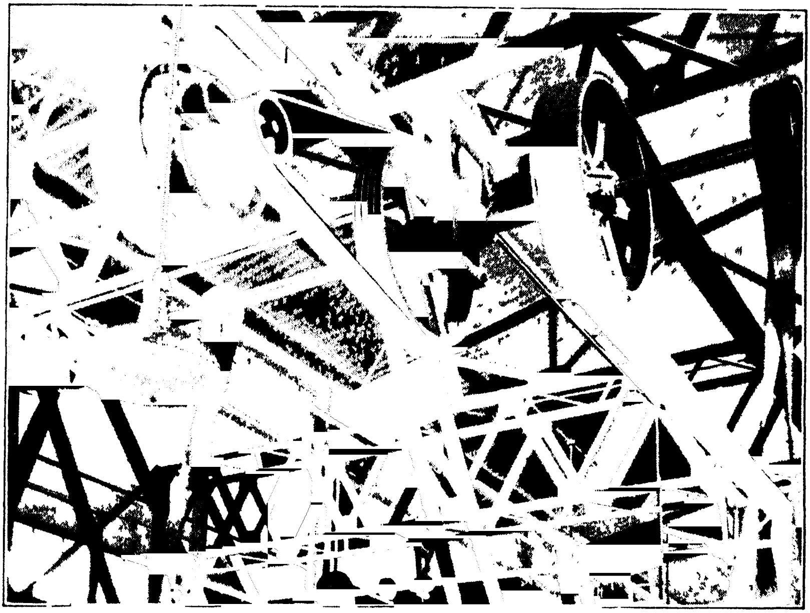


Fig. 67 Line Shaft with Both Open- and Crossed-Belt Drives
Courtesy of Dodge Manufacturing Corp., Mishawaka, Ind.

N_a = the r.p.m. of pulley A

N_b = the r.p.m. of pulley B

V_{La} = the linear velocity of the rim of pulley A

V_{Lb} = the linear velocity of the rim of pulley B

From formula (11)

$$V_{La} = 2\pi R_a N_a$$

and

$$V_{Lb} = 2\pi R_b N_b$$

Since

$$V_{La} = V_{Lb}$$

$$2\pi R_a N_a = 2\pi R_b N_b$$

Dividing both members of the equation by $2\pi R_a N_b$

$$\frac{2\pi R_a N_a}{2\pi R_a N_b}=\frac{2\pi R_b N_b}{2\pi R_a N_b}$$

Canceling like factors in each member of the equation

$$\frac{N_a}{N_b}=\frac{R_b}{R_a}=\text{a constant} \tag{27}$$

Also multiplying both numerator and denominator of the second member of formula (27) by 2,

$$\frac{N_a}{N_b}=\frac{2R_b}{2R_a}=\frac{D_b}{D_a}=\text{a constant} \tag{28}$$

From formulas (27) and (28) it may be stated that the r.p.m. ratio of driver to follower in a belt drive is constant, and is both inversely as the radii and inversely as the diameters of the pulleys.

From formulas (25) and (27) or formulas (26) and (28) it may be seen that the angular velocity ratio of driver to follower, $\frac{V_{aa}}{V_{ab}}$, and the r.p.m. ratio, $\frac{N_a}{N_b}$, are both equal to the same thing; therefore

$$\frac{V_{aa}}{V_{ab}}=\frac{N_a}{N_b} \tag{29}$$

The truth of this statement is self-evident when one considers that a revolution about an axis is in reality a unit of angular displacement. In fact in any of the modes of transmission of motion, the r.p.m. ratio is equivalent to the angular velocity ratio.

Note. Formulas (25), (26), (27), and (28) are applicable to V-belt and rope drives when the pitch radii or pitch diameters of their sheaves are used.

Example 1. If the diameters of pulleys A and B in either Fig. 68 or Fig. 69 are 32 inches and 8 inches respectively, and the r.p.m. of B is 800, what is the r.p.m. of A?

Solution. Here $D_a=32$ in., $D_b=8$ in., and $N_b=800$ r.p.m. Applying formula (28)

$$\frac{N_a}{N_b}=\frac{D_b}{D_a}$$

and evaluating therein

$$\frac{N_a}{800}=\frac{8}{32}=\frac{1}{4}$$

Multiplying both members of the equation by 800,

$$N_a = \frac{1}{4} \times 800 = 200 \text{ r.p.m. } \textit{Ans.}$$

Example 2. The drive shaft of a machine is belt-connected to the armature shaft of a motor which makes 1350 r.p.m. If a 10-inch pulley is used on the armature shaft, what size of pulley must be selected for the drive shaft of the machine in order that the latter shall have an r.p.m. of 300?

Solution. Let the pulley on the armature shaft be called A, and the pulley on the drive shaft, B. Then $D_a = 10$ in., $N_a = 1350$ r.p.m., $N_b = 300$ r.p.m.

Using formula (28),

$$\frac{N_a}{N_b} = \frac{D_b}{D_a}$$

and evaluating therein

$$\frac{1350}{300} = \frac{D_b}{10}$$

Multiplying both members of the equation by 10,

$$D_b = 10 \times \frac{1350}{300} = 45 \text{ in. } \textit{Ans.}$$

Directional Relation of Driver and Follower. An inspection of the open-belt drive of Fig. 68 clearly reveals that both driver and follower must have the same direction of rotation. This is due to the fact that the belt, in passing from one pulley to the other, does not pass between the fixed centers of the pulleys. On the other hand, in the crossed-belt drive of Fig. 69, the belt does pass between the fixed centers and it will be noted that the pulleys must rotate in opposite directions.

Thus the directional relation of driver and follower in any wrapping-connector mechanism is determined by merely noticing the manner in which the line of the connector (tT of Figs. 68 and 69) cuts the line of fixed centers of the mechanism. Hence directional relation is established in this mode of transmission of motion in exactly the same way as in the rigid-link connector mechanism. (See Chapter III.)

Length of Open Belt. Fig. 68 illustrates an open belt stretched over two pulleys, A and B, which are keyed to parallel shafts. It will be noted that the belt is tangent to the pulley rims at t and T, and hence is perpendicular to the radii Ot and O_1T, drawn to these points of tangency. Let the radius of the larger pulley, A, be denoted by R_a, and the radius of the smaller pulley, B, be denoted by R_b. The center of pulley A (or its shaft) is point O_1, while the center of pulley B is point O; and the distance between these centers is given in the figure by the dimension C.

If a line is drawn from center O parallel to tT, the rectangle $OsTt$ is formed, in which $Os = tT$. Also, the right triangle OsO_1 is formed, in which the hypotenuse OO_1 has a length equal to C, the distance between centers; and the acute angle O_1Os, called θ (theta), is equal to the central angle NOt of pulley B, and to the central angle MO_1T of pulley A. For two angles are equal when the sides of one are either parallel or perpendicular to the sides of the other.

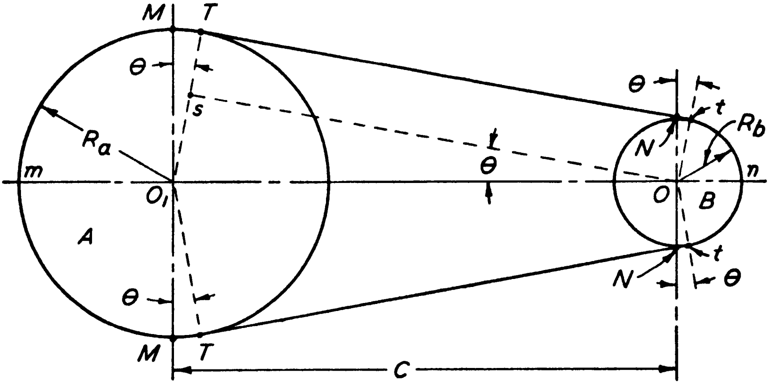


Fig. 68. Open-Belt Drive

Now, in the triangle OsO_1

$$O_1s = O_1T - sT$$

but $$O_1T = R_a$$

and $$sT = Ot = R_b$$

therefore $$O_1s = R_a - R_b$$

Also in the triangle OsO_1

$$\sin\theta = \frac{O_1s}{OO_1}$$

Since $$O_1s = R_a - R_b, \text{ and } OO_1 = C$$

$$\sin\theta = \frac{R_a - R_b}{C}$$

or $$\theta = \sin^{-1}\frac{R_a - R_b}{C}$$

Note. The above step is read as follows: θ is the angle whose sine is $\frac{R_a - R_b}{C}$.

An inspection of the figure indicates that the length L_o, of the open belt, is given by the statement,

$$L_o = 2tT + 2(\text{arc } MT) + \text{arc } MmM + \text{arc } NnN - 2(\text{arc } Nt)$$

In this statement,

$$tT = Os = \sqrt{\overline{OO_1}^2 - \overline{O_1s}^2} = \sqrt{C^2 - (R_a - R_b)^2}$$

(for one side of a right triangle is equal to the square root of the difference obtained by subtracting the square of the other side from the square of the hypotenuse).

$$\text{arc } MT = R_a\theta = R_a \times \sin^{-1}\frac{R_a - R_b}{C}$$

in which θ, or $\sin^{-1}\frac{R_a - R_b}{C}$ is in radians

arc $MmM = \pi R_a$, the half circumference of the larger pulley,

arc $NnN \ = \pi R_b$, the half circumference of the smaller pulley,

$$\text{arc } Nt = R_b\theta = R_b \times \sin^{-1}\frac{R_a - R_b}{C}$$

in which $\sin^{-1}\frac{R_a - R_b}{C}$ is in radians.

Substituting these values in the statement of L_o,

$$L_o = 2\sqrt{C^2 - (R_a - R_b)^2} + 2R_a \sin^{-1}\frac{R_a - R_b}{C} + \pi R_a + \pi R_b - 2R_b \sin^{-1}\frac{R_a - R_b}{C}$$

Rearranging terms,

$$L_o = 2\sqrt{C^2 - (R_a - R_b)^2} + \pi R_a + \pi R_b + 2R_a \sin^{-1}\frac{R_a - R_b}{C} - 2R_b \sin^{-1}\frac{R_a - R_b}{C}$$

Simplifying,

$$L_o = 2\sqrt{C^2 - (R_a - R_b)^2} + \pi(R_a + R_b) + 2(R_a - R_b)\sin^{-1}\frac{R_a - R_b}{C} \qquad \textbf{(30)}$$

A close approximation to the length of an open belt is given by the following formula,

$$L_o=2C+\frac{\pi}{2}(D_a+D_b)+\frac{(D_a-D_b)^2}{4C} \tag{31}$$

where D_a and D_b are the pulley diameters. This formula produces results of sufficient accuracy for most cases.

Length of Crossed Belt. Fig. 69 shows a crossed belt in use with pulleys A and B. In this case, the notation used is identical to that used in the open belt of Fig. 68. As in the preceding case, if the line Os is drawn from center O of the smaller pulley, parallel to tT, and terminating in radius O_1T, extended, of the larger pulley at point s, the rectangle $OsTt$ is formed, in which $Os=Tt$. Likewise the right triangle OsO_1 is formed, in which angle O_1Os, or $\theta, = \angle NOt = \angle MO_1T$ and hypotenuse $OO_1=C$, the distance between the centers of the pulleys. In this right triangle however

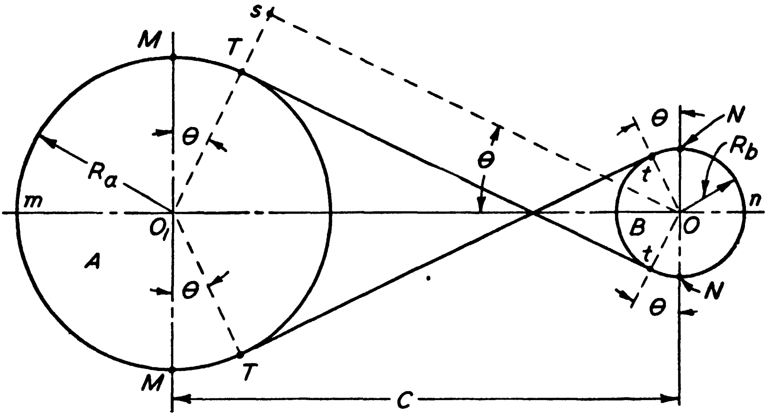


Fig. 69. Crossed-Belt Drive

$$O_1s=O_1T+Ts=R_a+R_b$$

since

$$Ts=Ot=R_b$$

and

$$\sin\theta=\frac{O_1s}{OO_1}=\frac{R_a+R_b}{C}$$

or

$$\theta=\sin^{-1}\frac{R_a+R_b}{C}$$

An inspection of Fig. 69 indicates that the length, L_c, of the crossed belt is given by the statement

$$L_c=2tT+2(\text{arc } MT)+\text{arc } MmM+\text{arc } NnN+2(\text{arc } Nt)$$

In this statement,

$$tT = Os = \sqrt{\overline{OO_1}^2 - \overline{O_1s}^2} = \sqrt{C^2 - (R_a+R_b)^2}$$

$$\text{arc } MT = R_a\theta = R_a \times \sin^{-1}\frac{R_a+R_b}{C}$$

in which θ, or $\sin^{-1}\frac{R_a+R_b}{C}$, is expressed in radians

arc $MmM = \pi R_a$, the half circumference of the larger pulley,
arc $NnN = \pi R_b$, the half circumference of the smaller pulley,

$$\text{arc } Nt = R_b\theta = R_b \times \sin^{-1}\frac{R_a+R_b}{C}$$

in which $\sin^{-1}\frac{R_a+R_b}{C}$ is expressed in radians.

Substituting these values in the statement of L_c,

$$L_c = 2\sqrt{C^2 - (R_a+R_b)^2} + 2R_a \sin^{-1}\frac{R_a+R_b}{C} + \pi R_a + \pi R_b + 2R_b \sin^{-1}\frac{R_a+R_b}{C}$$

Simplifying,

$$L_c = 2\sqrt{C^2 - (R_a+R_b)^2} + \pi(R_a+R_b) + 2(R_a+R_b)\sin^{-1}\frac{R_a+R_b}{C} \qquad (32)$$

Formulas (**30**), (**31**), and (**32**) show that the length of either an open belt or a crossed belt can be obtained if only the radii of the pulleys and the distance between their centers are known. In the application of each of these formulas, the same unit of length must be used of course for each radius, as well as for the distance between centers. This unit then becomes the unit in which the length of the belt will be expressed.

Belt lengths for both crossed and open belts are obtained frequently by measuring a cord placed over the pulleys when installed, or by the measurement of a scale drawing.

Example 1. It is required to obtain the length of an open belt to be used to connect a 12-inch pulley to a 40-inch pulley, the distance between their centers being 5 feet.

Solution. Here the radius of the larger pulley, $R_a = \frac{40}{2} = 20$ in.; the radius of the smaller pulley, $R_b = \frac{12}{2} = 6$ in., and $C = 5 \times 12 = 60$ in. Applying formula (**30**),

$$L_o = 2\sqrt{C^2 - (R_a-R_b)^2} + \pi(R_a+R_b) + 2(R_a-R_b)\sin^{-1}\frac{R_a-R_b}{C}$$

and evaluating therein

$$L_o=2\sqrt{60^2-(20-6)^2}+\pi(20+6)+2(20-6)\sin^{-1}\frac{20-6}{60}$$

The procedure for finding $\sin^{-1}\frac{20-6}{60}$ in the radian as a unit, is as follows:

$$\sin^{-1}\frac{20-6}{60}=\sin^{-1}\frac{14}{60}=\sin^{-1}0.2333$$

From a table of natural sines,

$$\sin^{-1}0.2333=13°30'=13.5°$$

Changing from the degree to the radian as a unit,

$$13.5°=13.5\times\frac{2\pi}{360}=0.2356 \text{ radians}$$

Therefore

$$\sin^{-1}0.2333=0.2356 \text{ radians}$$

Substituting this value in the equation and simplifying,

$$L_o=2\sqrt{3404}+26\pi+28\times0.2356$$

$$L_o=116.69+81.68+6.60$$

Collecting terms of the second member

$$L_o=204.97 \text{ in.} \quad \textit{Ans.}$$

Example 2. Solve the preceding example for the approximate length of open belt afforded by formula (**31**).

Solution. Here $D_a=40$ in., $D_b=12$ in., and $C=60$ in.

Applying formula (**31**),

$$L_o=2C+\frac{\pi}{2}(D_a+D_b)+\frac{(D_a-D_b)^2}{4C}$$

and evaluating therein,

$$L_o=2\times60+\frac{\pi}{2}(40+12)+\frac{(40-12)^2}{4\times60}$$

$$L_o=120+81.68+3.27$$

Collecting,

$$L_o=204.95 \text{ in.} \quad \textit{Ans.}$$

Example 3. It is required to obtain the length of a crossed belt to be used with the mechanism of the preceding example.

Solution. Here $R_a=20$ in., $R_b=6$ in., and $C=60$ in.

Applying formula (**32**),

$$L_c=2\sqrt{C^2-(R_a+R_b)^2}+\pi(R_a+R_b)+2(R_a+R_b)\sin^{-1}\frac{R_a+R_b}{C}$$

and substituting therein

$$L_c=2\sqrt{60^2-(20+6)^2}+\pi(20+6)+2(20+6)\sin^{-1}\frac{20+6}{60}$$

$$L_c=2\sqrt{2924}+26\pi+52\times\sin^{-1}\frac{26}{60}$$

Solving for $\sin^{-1}\frac{26}{60}$

$$\sin^{-1}\frac{26}{60}=\sin^{-1}0.4333=25°41'$$

$$25°41'=25\frac{41°}{60}=25\frac{41}{60}\times\frac{2\pi}{360}$$

$$=0.4483 \text{ radians}$$

$$L_c=2\sqrt{2924}+26\pi+52\times0.4483$$

$$L_c=108.15+81.68+23.31$$

Collecting

$L_c=213.14$ in. *Ans.*

The Law of Belting. As a belt runs over its pulleys, each of its two sides leaves or recedes from one pulley and approaches the other. Thus in Fig. 66 with the direction of motion as shown, the lower side of the belt is the approaching side with respect to pulley *A*, while it is the receding side with respect to pulley *B*. Likewise the upper side of the belt is the approaching side with respect to *B*, and the receding side with respect to *A*. (Also see Fig. 70.)

In order that a belt will not shift along the rim of its pulleys and tend to run off the pulleys, its installation must be in accordance with a certain principle which is frequently called the Law of Belting. This law states that the center line of the approaching side of the belt must lie in the mid, or center, plane of the pulley over which the belt is to advance. The center plane of the pulley is a plane which cuts through the midsection of the pulley perpendicular to its axis.

With parallel shafts, the center plane of the driver is coincident with the center plane of the follower. For this reason, the center line of one side of the belt lies in the center planes of both pulleys simultaneously. Hence in this case the direction of motion of the belt may be reversed and the law of belting will continue to be adhered to. Therefore with parallel shaft installations, the belt may be run in either direction without shifting axially along its pulleys. That this is not possible with nonparallel shafts will be shown later in this chapter.

Shifting of Belts. A flat belt may be shifted to a new position on its pulley by introducing a force on the edge of the approaching side of the belt and relatively close to that pulley over which the belt is to advance. This is shown in Fig. 70 where the force *F* is directed in such a manner that it will move the belt to the left along its pulley. This force causes the belt to take the new position shown by the dotted lines. The successive portions of the belt as displaced by the force are then carried by the pulley in a plane perpendicular to its axis, until finally the belt is in its new position with the center line of its approaching side lying in a plane perpendicular to the axis of the pulley. Should the force *F* be introduced on the opposite side (the receding side) of the pulley, it would have little effect unless it is large enough to overcome the frictional resistance between the belt and the pulley.

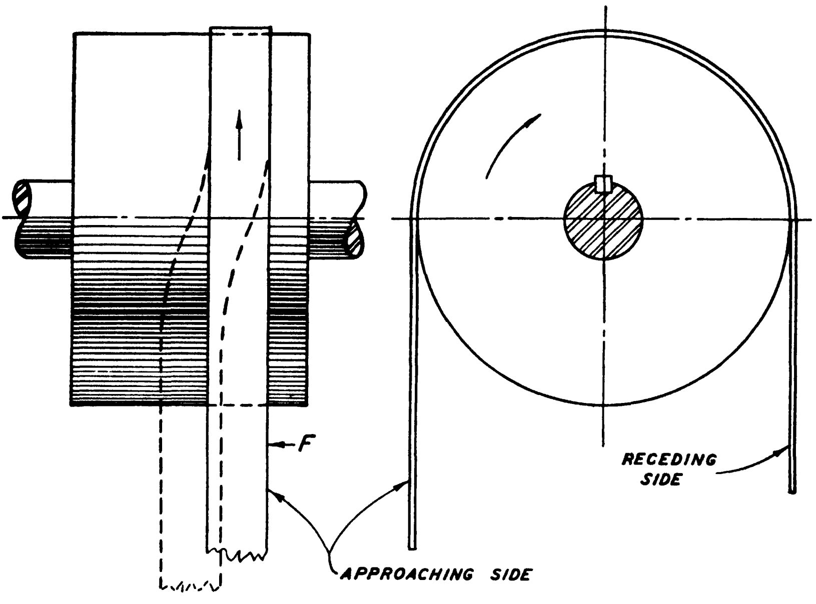


Fig 70 Belt Shifting

A mechanism for shifting a belt consists of a rod to which is attached a pair of fingers that span the belt on the approaching side. The rod, mounted parallel to the pulley axis, may be pulled back and forth by a hand lever so as to shift the belt in either direction.

Crowning of Pulleys. Due to lack of alignment or imperfections in the construction of the pulleys, the pull on a belt may be greater on one of its edges than on the other. In such a case, the belt in motion will shift laterally along the faces of the pulleys toward that edge at which the pull is the greater. To forestall this tendency of a belt to run off, the thickness of a pulley rim is increased at the center as shown in Fig. 71 (*a*) and (*b*). A pulley with such a rim is said to be *Crowned.* The crowning of a pulley gives it a larger diameter at its midplane than at either edge, so that if the belt is off center it will shift towards the midplane of the pulley as the pulley rotates. When the center line of the belt has been thus shifted so that it lies in the midplane of the pulley, the belt has an equal tendency to shift in each direction, therefore it is held in a central position on its pulley.

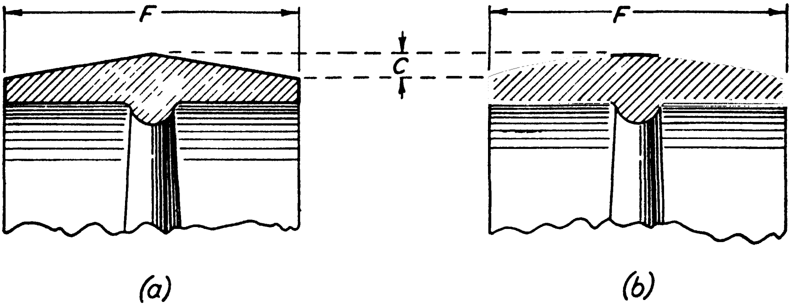


Fig 71 Rims of Crowned Pulleys

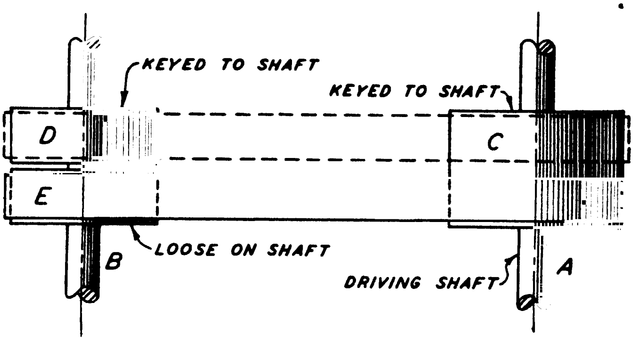


Fig. 72. Tight and Loose Pulleys

Pulleys are crowned either by giving the rim a straight taper on each side of the midplane as shown in Fig. 71 (*a*) or by rounding off the rim surface as shown in Fig. 71 (*b*). The latter is the better method, but as it is more costly, the former is generally employed. In general, the taper that provides the height of the crown, *C*, should not be greater than ⅛ inch per foot of face *F*.

Tight and Loose Pulleys. Fig. 72 shows a very common arrangement of pulleys whereby the driven shaft, *B*, may be stopped while the driving shaft, *A*, continues to rotate. Pulley *D*, the so-called tight pulley, is keyed to shaft *B*. Pulley *E*, the so-called loose pulley, is free to run on shaft *B*. Pulley *C*, the driver, has its face equal in width to the sum of the faces of the other two, plus the clearance between them, and is keyed to the driving shaft *A*. When the belt is in the position shown by the full lines, pulley *E* turns with pulley *C*, but shaft *B* remains at rest. When it is desired to start shaft *B*, a belt shipper shifts the belt to the dotted position, in which *C* then drives the tight pulley *D*.

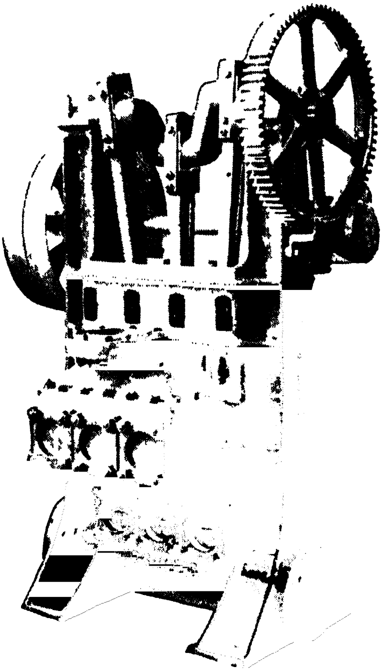


Fig 73 Tight and Loose Pulleys Installed on a Goulds Triplex Pump

Courtesy of The Goulds Manufacturing Co Seneca Falls, N Y

The tight and loose pulleys, *D* and *E*, are both crowned. The driving pulley, *C*, is not crowned, which permits shifting the belt more easily. The hub of the loose pulley is made longer than that of the tight pulley, since the former furnishes a bearing surface with the driven shaft. The diameter of the loose pulley is often made slightly smaller than the diameter of the tight pulley in order to relieve the belt tension during the period when no power is being transmitted.

An illustration of a tight and loose pulley installation is shown in Fig. 73

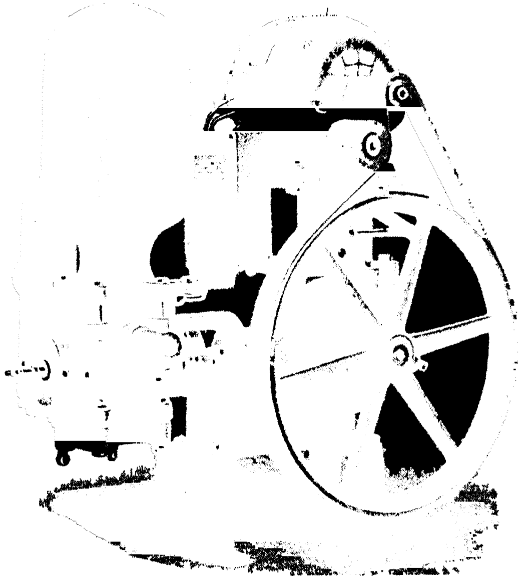


Fig 74 Tension Pulley in Use with a Flat-Belt Drive
Courtesy of Deming Co Salem Ohio

Idler or Tension Pulleys. It often happens in a belt drive that the driving and driven pulleys need be given radically different diameters in order to secure a desired r.p.m. ratio, and that they also must be placed close together. These conditions, occurring simultaneously, produce a very small arc of contact between the belt and the smaller pulley. This in turn materially reduces the power that may be transmitted. In addition to this it is difficult, in the short belt which is necessarily used, to maintain the correct tightness or tension. These difficulties can be overcome by the use of a third pulley as shown in Fig. 74. It will be noted that this pulley, called an *Idler* or *Tension*

Pulley, is so placed that it increases the arc of contact. It is capable of adjustment, in its position relative to the belt, so that the proper tension in the belt can be maintained.

An idler pulley is sometimes used as a *Guide Pulley*; that is, its function is to guide the belt so that the center line of the belt, as the latter approaches a pulley, lies in the midplane of the latter. Such a guide pulley is sometimes necessary in a belt drive between nonparallel shafts.

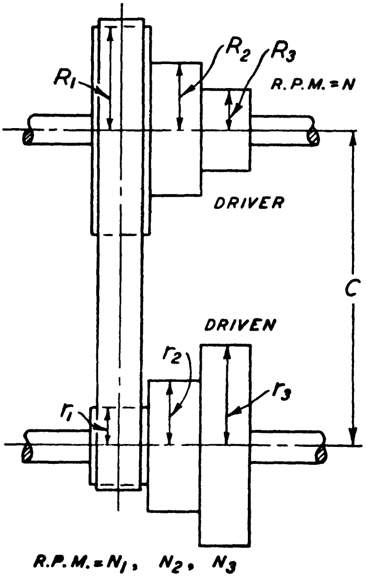


Fig. 75. Cone or Stepped Pulleys

Cone or Stepped Pulleys. A *Cone Pulley* is one whose rim has several different diameters. It is in reality a series of pulleys placed side by side on a shaft and operating as a single unit. Such a pulley is also known as a *Stepped Pulley* due to the step-like appearance of its rim. When two such pulleys are used in a belt drive, the smallest step of one is placed in line with the largest step of the other, as shown in Fig. 75. Since the belt can be used with any pair of steps, it is evident that there will be as many angular velocity ratios possible between the driving and driven pulleys as there are steps on each pulley. In other words, for a constant speed of the driver, the driven shaft can have as many different speeds as there are pairs of steps.

Hence in Fig. 75, the driven shaft, or its stepped pulley, can have three speeds. With the belt in the position shown in the figure, it is evident from formulas (27) and (28) that the driven shaft will receive its maximum angular velocity or r.p.m. The cone pulleys of Fig. 75 are equal in size; this however is not always the case.

Since the same belt is employed with each pair of steps, the radii of corresponding steps must be so selected that they will not only provide a required angular velocity ratio, but will also be suited to the same length of belt as all other pairs of steps. Cone pulleys are generally used with open belts because it is easier to shift an open belt than one that is crossed; crossed belts, however, are sometimes used. The design of cone pulleys for each type of belt installation will be considered.

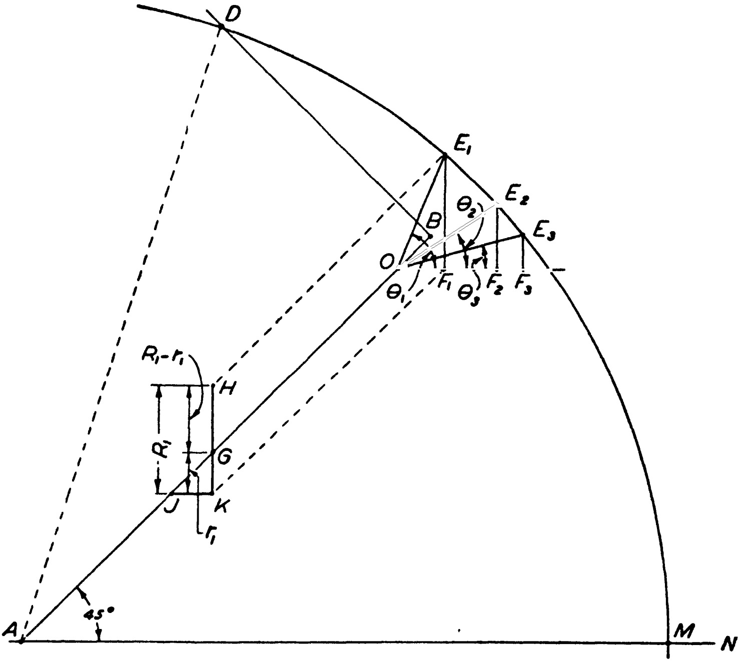


Fig. 76. Burmester's Method

Design of Cone Pulleys Using an Open Belt. A mathematical solution of this problem is rather involved, due to the fact that both the sum and difference of a pair of corresponding radii occur in the formula for the length of an open belt. See formula (30). However, Dr. L. Burmester of Leipzig, Germany, has derived a graphical method of solution which is extremely accurate. This method is illustrated in Fig. 76. In effecting this solution for the radii of the steps, it is necessary to know the constant speed in r.p.m. of the driving shaft, the desired speed in r.p.m. of the driven shaft, and the distance between centers of shafts.

As in Fig. 75, let

N = the speed of the driver in r.p.m.

N_1, N_2, N_3 = the required speeds of the driven, in which N_1 is the maximum speed in r.p.m.

C = the distance between centers, in inches

R_1, R_2, R_3 = radii of the driver, in inches

r_1, r_2, r_3 = radii of the driven, in inches

Before proceeding with the graphical solution of Fig. 76, it is necessary to find the radii of a pair of corresponding steps. For example, the radii of the left-hand pair of steps of Fig. 75 will be found. Since N (the constant speed of the driver) and N_1 (the maximum speed of the driven or follower) are known, the radius R_1 of the largest step of the driver may be assumed and the unknown radius, r_1, may be found by applying formula (27).

Thus $$\frac{R_1}{r_1}=\frac{N_1}{N}$$

or $$r_1=\frac{N}{N_1}\times R_1$$

Having selected a scale that will give a drawing of rather large size, Fig. 76, lay off an indefinite horizontal base line AN, and draw line AB at an angle of 45 degrees to AN, making AB equal in length to the distance between centers, C. Draw a perpendicular to AB at point B and lay off on this perpendicular a distance BD, equal to $\frac{C}{2}$. With A as a center and the distance AD as a radius, describe an indefinite arc, DM. Through any point on line AB, such as point G, draw a vertical line and lay off on this vertical line a distance

$$GH=R_1-r_1$$

and also a distance $$GK=r_1$$

Then $$HK=HG+GK$$

$$=(R_1-r_1)+r_1=R_1$$

Draw a horizontal line through K intersecting AB at J. Then triangle GKJ is an isosceles right triangle. Therefore

$$JK = GK = r_1$$

Now through H draw a line parallel to AB intersecting arc DM at E_1. Through K draw a line KF_1 parallel to AB. Draw a vertical line through E_1 intersecting KF_1 at F_1. Polygon HE_1F_1K is a parallelogram; therefore

$$E_1F_1 = HK = R_1$$

Through F_1 draw a horizontal line intersecting AB at point O. Polygon JOF_1K is also a parallelogram; therefore

$$OF_1 = JK = r_1$$

Connect O to E_1 forming the right triangle OF_1E_1, in which the acute angle F_1OE_1 is named θ_1. Then

$$\tan\theta_1 = \frac{E_1F_1}{OF_1} = \frac{R_1}{r_1}$$

But, as previously seen,

$$\frac{R_1}{r_1} = \frac{N_1}{N}$$

therefore $$\tan\theta_1 = \frac{N_1}{N}$$

or $$\theta_1 = \tan^{-1}\frac{N_1}{N}$$

Now in the second pair of steps of Fig. 75,

$$\frac{R_2}{r_2} = \frac{N_2}{N}$$

let $$\theta_2 = \tan^{-1}\frac{N_2}{N}$$

Since in this step N_2 and N are known values, the angle θ_2 can be determined. In the drawing (see Fig. 76) lay off line OE_2, intersecting arc DM at E_2 and making angle θ_2 with the horizontal line through point O. Draw a vertical line through E_2 that intersects OF_1 (extended) at F_2.

Then $$\theta_2 = \tan^{-1}\frac{E_2F_2}{OF_2}$$

but $$\theta_2 = \tan^{-1}\frac{N_2}{N} = \tan^{-1}\frac{R_2}{r_2}$$

Therefore $$\tan^{-1}\frac{R_2}{r_2} = \tan^{-1}\frac{E_2F_2}{OF_2}$$

and
$$\frac{R_2}{r_2}=\frac{E_2F_2}{OF_2}$$
so that the radius of the second step of the driver is

$R_2=E_2F_2$ (multiplied by the scale of the drawing)

and the radius of the second step of the driven is

$r_2=OF_2$ (multiplied by the scale of the drawing)

That this pair of radii, R_2 or E_2F_2 and r_2 or OF_2, will give the required speed ratio, $\frac{N_2}{N}$, is evident from the construction of the drawing.

In a similar manner, R_3 and r_3 can now be found by introducing into Fig. 76 an angle
$$\theta_3=\tan^{-1}\frac{R_3}{r_3}=\tan^{-1}\frac{N_3}{N}$$
For, from formula (27), for the third or right-hand pair of steps of Fig. 75
$$\frac{R_3}{r_3}=\frac{N_3}{N}$$
Having found θ_3 and introduced it into Fig. 76,

$R_3=E_3F_3$ (multiplied by the scale of the drawing)

$r_3=OF_3$ (multiplied by the scale of the drawing)

When all pairs of radii have been found, each pair in turn should be substituted, along with C, in formula (30) to see if the same length of belt is obtained. The lengths of belt should check very closely, otherwise there is an error in the solution.

Design of Cone Pulleys Using a Crossed Belt. An inspection of formula (32), which gives the length of a crossed belt, discloses the fact that in every term of that formula the sum of the radii R_a and R_b is always involved. This indicates that different pairs of radii having a constant sum, when introduced in formula (32), will produce the same length of crossed belt if the same distance between centers, C, is used. For instance, suppose a crossed belt is used with a pair of three-step cone pulleys similar to those of Fig. 75, and let it be further supposed that
$$R_1+r_1=R_2+r_2=R_3+r_3$$
It is evident that no matter which one of these pairs of radii are introduced into formula (32) for (R_a+R_b), the resulting length of belt, L_c, will be the same. Therefore with a pair of cone pulleys using a crossed belt, it is necessary simply to have the sum of the radii of each pair of

steps remain constant. Of course each pair of radii must be selected so that they will produce the required angular velocity ratio; in other words, formula (27) or (28) must be satisfied.

Example. Two equal cone pulleys having three steps are used to connect two shafts, A and B, which are 4 feet between centers. The radii of the steps are 2 inches, 4 inches, and 6 inches. The driving shaft, A, makes 180 revolutions per minute. The shafts rotate in opposite directions, which necessitates the use of a crossed belt.

(*a*) Will these cone pulleys permit the use of the same belt with each pair of steps?

(*b*) Obtain the speeds of the driven shaft, B.

(*c*) What length of belt will be used?

Solution. (*a*) Using the notation of Fig. 75, the radii of the pulley on the driving shaft are as follows:

$$R_1=6 \text{ in.},\ R_2=4 \text{ in.},\ R_3=2 \text{ in.}$$

Since in using two equal cone pulleys the steps are reversed, the radii of the pulleys on the driven shaft are as follows:

$$r_1=2 \text{ in.},\ r_2=4 \text{ in.},\ r_3=6 \text{ in.}$$

Therefore the sums of the radii of corresponding steps are:

$$R_1+r_1=6+2=8 \text{ in.}$$

$$R_2+r_2=4+4=8 \text{ in.}$$

$$R_3+r_3=2+6=8 \text{ in.}$$

Since the sum of the radii remains constant, a crossed belt may be used. *Ans.*

(*b*) $N=180$ r.p.m., the constant speed of the driver. N_1, N_2, and N_3 = the speeds of the driven shaft in order from left to right in Fig. 75.

Applying formula (27) to the left-hand pair of steps,

$$\frac{N_1}{N}=\frac{R_1}{r_1}$$

and evaluating therein,

$$\frac{N_1}{180}=\frac{6}{2}$$

$$N_1=180\times\frac{6}{2}=540 \text{ r.p.m.} \quad \textit{Ans.}$$

Applying formula (27) to the middle pair of steps,

$$\frac{N_2}{N}=\frac{R_2}{r_2}$$

and evaluating therein,

$$\frac{N_2}{180}=\frac{4}{4}$$

$$N_2=180\times\frac{4}{4}=180 \text{ r.p.m.} \quad \textit{Ans.}$$

Applying formula (27) to the right-hand pair of steps,

$$\frac{N_3}{N}=\frac{R_3}{r_3}$$

and evaluating therein,

$$\frac{N_3}{180}=\frac{2}{6}$$

$$N_3=180\times\frac{2}{6}=60 \text{ r.p.m.} \quad \textit{Ans.}$$

(*c*) Since all pairs of steps will use the same belt, the belt length may be found by using the radii of the left-hand pair. Hence,

$$R_1=6 \text{ in.}, r_1=2 \text{ in.}, C=4\times12=48 \text{ in.}$$

Applying formula (32)

$$L_c=2\sqrt{C^2-(R_1+r_1)^2}+\pi(R_1+r_1)+2(R_1+r_1)\sin^{-1}\frac{R_1+r_1}{C}$$

and evaluating therein

$$L_c=2\sqrt{48^2-(6+2)^2}+\pi(6+2)+2(6+2)\sin^{-1}\frac{6+2}{48}$$

$$L_c=2\sqrt{2240}+8\pi+16\sin^{-1}0.1667$$

From a table of natural sines

$$\sin^{-1}0.1667=9°36'=9.6°$$

Changing the angle, $\sin^{-1}0.1667$, to radian measure by multiplying by $\frac{2\pi}{360}$

$$9.6°=9.6\times\frac{2\pi}{360}=0.1676 \text{ radians}$$

Substituting this value in our equation,

$$L_c=2\sqrt{2240}+8\pi+16\times0.1676$$

$$L_c=94.66+25.13+2.68$$

Collecting, $L_c=122.47$ in. *Ans.*

Nonparallel Shaft or Twisted-Belt Drives. Although belts are most often employed to connect parallel shafts, they may also be used to connect shafts which are nonparallel. This is demonstrated by Fig. 77 (*a*) and (*b*).

In Fig. 77(*a*), two pulleys, *A* and *B*, are located on parallel shafts. The center line of the belt is tangent to pulley *A* at points a_1 and a_2, and tangent to pulley *B* at points b_1 and b_2. The center line of that side of the belt which approaches or advances over pulley *B* is a_1b_1. (Note carefully the direction of motion as given in the figure.)

Therefore, points a_1 and b_1 lie in the central or mid-plane of *B*. The center line of that side of the belt which approaches pulley *A* is b_2a_2. Therefore, points a_2 and b_2 lie in the central plane of *A*. Of course with the parallel shafting of this figure, the central planes of the two pulleys are coincident. Now let the line S_1S_2 be drawn from a_1 to b_2, at which points the center line of the belt leaves the pulleys. Next pulley *B* and its shaft are turned about line S_1S_2 as an axis, to the position shown in Fig. 77(*b*). In this new position, S_1S_2 is the line of intersection of the central or mid-planes of pulleys *A* and *B*. Points a_1 and b_1 still lie in the mid-plane of *B*. Hence a_1b_1, the center line of the belt as it approaches pulley *B*, still lies in the mid-plane of the latter. Therefore the law of belting is satisfied and the belt will not tend to run off the pulleys as long as the given direction of motion is maintained. If, in Fig. 77(*b*), the pulleys are rotated in the opposite direction, the belt will run off immediately because reversal of motion will cause the approaching side for one pulley to become the approaching side for the other, and the law of belting is violated.

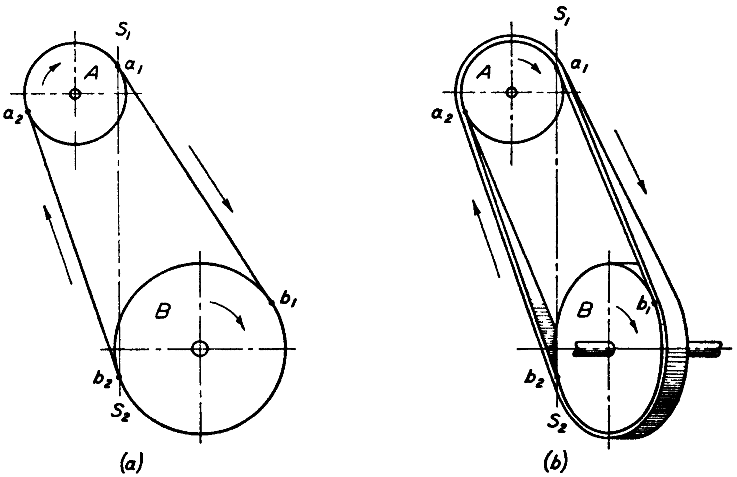


Fig. 77 Twisted-Belt Drive

A belt which is used to connect two nonparallel shafts, as in Fig. 77(*b*), is called a *Twisted Belt.* If pulley B is turned about line S_1S_2 until its axis is at 90 degrees to the axis of pulley A, the drive is said to be a Quarter-turn drive.

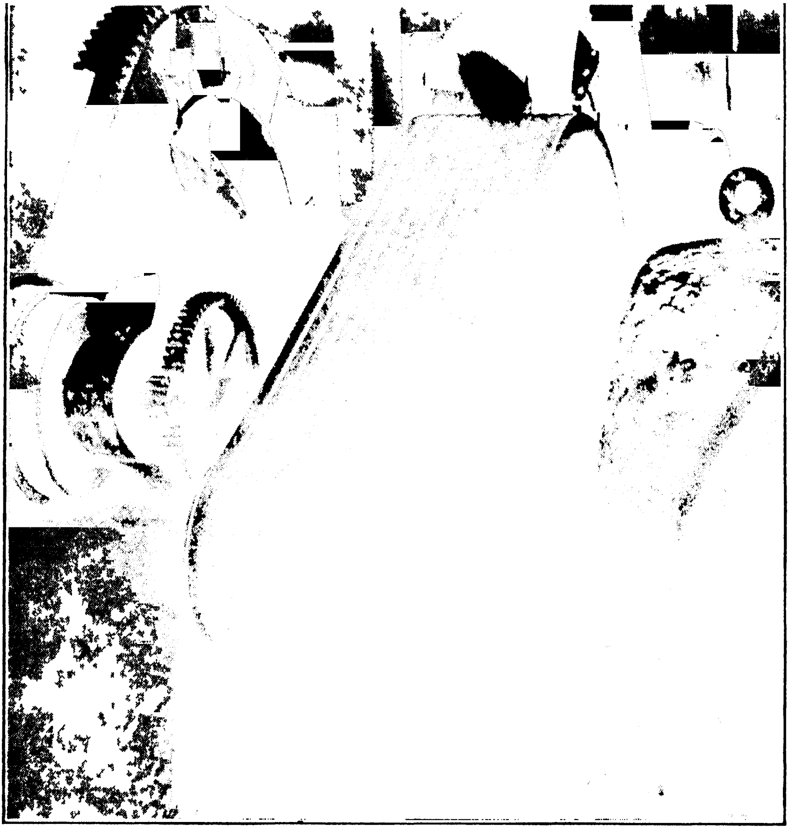


Fig 78 Typical V-Belt Drive
Courtesy of The American Pulley Co Philadelphia, Pa

V-Belt. A **V**-belt is a flexible connector that derives its name from the trapezoidal form of its cross section. As shown in Figs. 78 and 79, a belt of this type runs in a **V**-shaped groove on the pulley so that contact with the latter is made at the side of the belt. This produces a wedge-like action between the belt and the pulley. This wedge-like action is responsible for most of the favorable characteristics of the **V**-belt drive. Among these characteristics are:

1. Practically no slipping between the belt and its sheaves
2. Almost no deviation from a constant angular velocity ratio
3. Close hook-ups (small center distances between driver and follower) are feasible
4. Relatively high velocity ratios can be obtained
5. A considerable horsepower can be transmitted, even though the arc of contact between the **V**-belt and the smaller sheave is rather small.

V-belts may operate individually, as in the fan-belt drive of an automobile, or collectively as a multiple-drive by which as much as 500 horsepower can be transmitted. The installation of Fig. 78 is a multiple-drive.

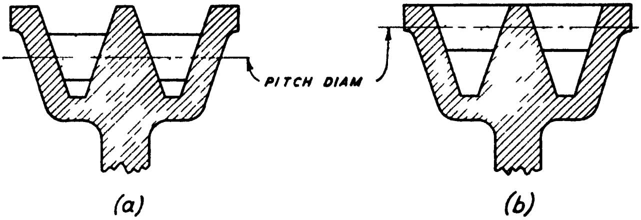


Fig. 79 Section of V-Belt Sheave, Showing that its Pitch Diameter Depends on Belt Used

Pitch Diameter of V-Belt Sheave. Within certain limits, a **V**-belt sheave can be used with various stock sizes of **V**-belts in, of course, different installations. This is illustrated in Fig. 79 (*a*) and (*b*), in which the smaller stock size of belt is used in Fig. 79(*a*), while a larger stock size is used with the same sheave in Fig. 79(*b*). Naturally, the smaller size rides much lower in the groove than the larger size.

Since the pitch diameter of a grooved pulley is taken at the mid-section or pitch surface (see Fig. 66) of its belt, the position in which the belt rides in the groove must be known before the pitch diameter can be ascertained. Since the pitch diameter, *D*, is involved in formulas dealing with velocities and velocity ratios, the latter depend on the position which the belt takes in the groove of its sheave.

Variable-Speed Belt Transmission. A variable-speed belt transmission is a mechanism that can produce a variety of speed ratios between the driving and driven shafts; so that, for a constant speed of driver, the driven shaft may have several different speeds.

The Reeves Pulley Company of Columbus, Indiana, makes use of the idea discussed in the preceding article in the design of their Variable-Speed Transmission. The latter is illustrated in Fig. 80, and its double-block V-belt is shown in Fig. 81. Each pulley of this transmission is in reality a pair of separate conical disks. The lateral or axial movement of the members of each pair dictates the position in which a V-belt rides (or is seated) upon them, and hence different pitch diameters may be obtained.

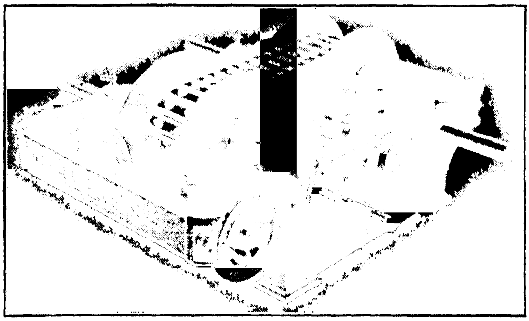


Fig 80 Reeves Variable-Speed Transmission, Enclosed Design with Cover Removed

Courtesy of Reeves Pulley Co , Columbus, Ind

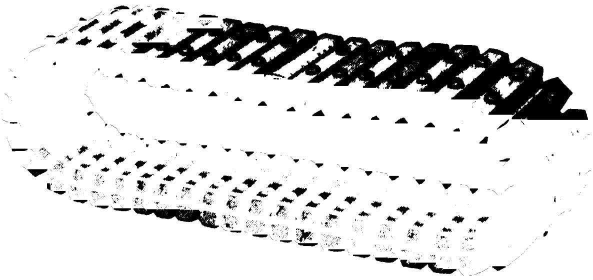


Fig 81 Double-Block V-Belt for Variable-Speed Transmission

Courtesy of Reeves Pulley Co , Columbus, Ind

Referring to Fig. 80, two pairs of cone-shaped disks are mounted on parallel shafts, and are connected by a V-shaped belt fitting the groove formed by the disks. The disks are adjusted laterally by two pairs of levers operated by a speed-changing screw. Rotation of this screw separates the levers and disks at one end of the transmission, reducing the pitch diameter that the V-belt takes on this set of disks. At the same time, the rotation of this screw brings together the levers and disks at the opposite end to increase the pitch diameter there. Thus when one shaft is driven at constant speed from any source of power, the speed of the opposite shaft is increased or decreased above and below the speed of the constant-speed shaft as the V-belt runs over different diameters of the disks. Infinitely adjustable speeds over a ratio of variation ranging from 2:1 to 16:1 are obtained through the use of this unit.

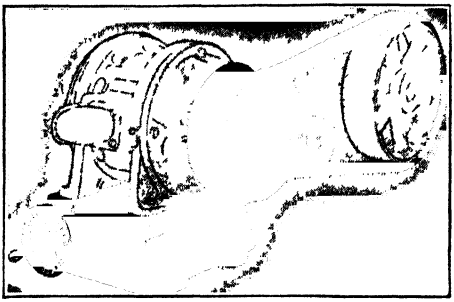


Fig 82 Reeves Vari-Speed Motor Pulley
Courtesy of Reeves Pulley Co, Columbus Ind

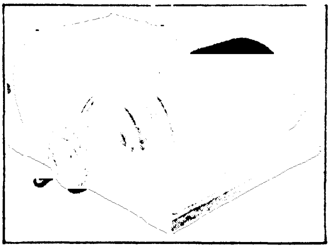


Fig 83 Reeves Motodrive, Horizontal Design without Reduction Gears
Courtesy of Reeves Pulley Co, Columbus, Ind

Either shaft of this transmission may be connected to the source of power and used as the constant-speed (power input) shaft, with the opposite shaft, driven at variable speeds, connected to the driven machine. The shafts may rotate either clockwise or counterclockwise.

The special design of V-belt used with the transmission unit shown in Fig. 81 has an endless-cord type body which is accurately clamped between two evenly spaced rows of hardwood blocks. These blocks are beveled and grooved to fit the V-groove between the disks.

The blocks are tipped with long-wearing specially tanned leather, tooled to conform to the convex surfaces of the disks.

The Reeves Pulley Company adapts the principle of their variable-speed transmission to two other units, which are their Vari-Speed Motor Pulley shown in Fig. 82 and their Vari-Speed Motodrive shown in Fig. 83.

Rope Drives. Although rope drives have been superseded to a great extent by other methods of transmitting power, certain conditions arise which still make a rope drive feasible and economical. This is particularly true when a considerable horsepower is to be transmitted between a pair of shafts whose distance between centers is rather large, say from 50 to 100 feet or more.

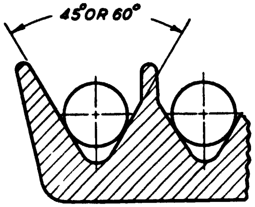


Fig 84 Textile Rope Laboring Sheave

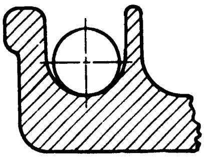


Fig 85 Textile Rope Idler Sheave

Both manila and cotton rope are used for power transmission. The sheaves with which they are used are pulleys with grooved rims similar to those used with **V**-belts. The grooves are finished smooth so as to minimize the external wear that would result from contact with a rough surface. A section of the rim of a rope sheave is shown in Fig. 84. The grooves not only hold the rope on the sheave, but also cause a wedge-like action to take place, thus increasing the power that can be transmitted by producing a greater frictional resistance between the rope and the sheave. Since this wedge-like action is to some extent destructive, a sheave when used only as an idler or tension sheave is grooved as in Fig. 85.

Systems of Rope Drive. There are two systems of rope driving. One is known as the *Multiple* or *English* system, and the other as the *Continuous* or *American* system.

The multiple system consists of several independent ropes running side by side in the grooves of the sheaves in a manner similar to the way in which **V**-belts are used in a multiple drive. Each rope is supposed to transmit the same amount of power. This requires that each rope be under the same initial tension, a condition that is difficult to obtain and hence a disadvantage of this system. However, a distinct advantage is that if one rope breaks, the drive can continue to function. This system is often used to distribute power to different machines from the same driving sheave.

The continuous system makes use of only one rope, which is wound around the driving and driven sheaves several times. In this system, Fig. 86, the rope is conducted from the outer groove on one side of the sheave to the inner or first groove by means of a travelling tension carriage or jockey. This carriage also serves to maintain a uniform tension throughout the rope. It is arranged to travel back and forth, automatically regulating the slack. In this manner the stretch in the rope and the inequalities in the load are taken care of, a distinct advantage of this system.

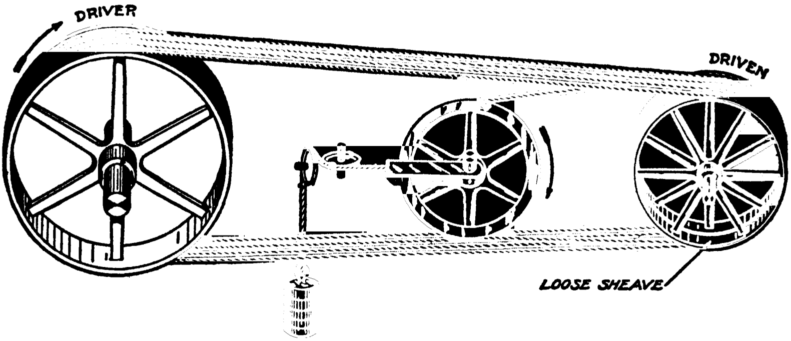


Fig. 86 Continuous Rope Drive, American System

Chain Drives. In all of the previous drives mentioned in this chapter, power is transmitted from one shaft to another through a frictional resistance which is set up between the wrapping connector and its pulley. In such cases slipping occurs to at least some small extent. This interferes with the angular velocity ratio between the driver and its follower. When it is imperative that the angular velocity ratio be constant, some form of so-called positive drive must be employed. The chain is that type of a wrapping connector which secures positive driving through its use with toothed wheels called *Sprockets.* A chain drive is used in preference to gears when the distance between centers of the shafts to be connected makes a gear drive unfeasible. With a chain drive, the mean angular velocity or r.p.m. ratio between driver and follower is constant and is inversely as the numbers of teeth on the sprockets; that is, if

N_1 = the r.p.m. of the driver
N_2 = the r.p.m. of the follower
t_1 = the number of teeth on the driver
t_2 = the number of teeth on the follower

then from the preceding statement,

$$\frac{N_1}{N_2}=\frac{t_2}{t_1} \tag{33}$$

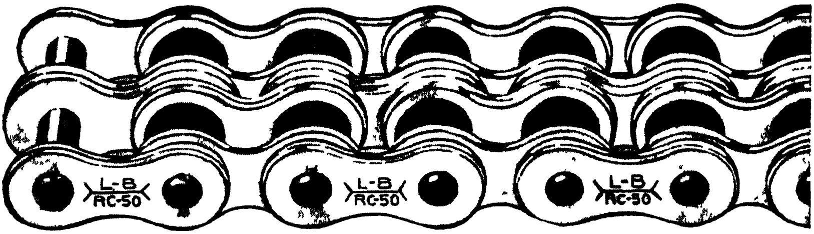


Fig 87 Double Roller Chain
Courtesy of Link-Belt Company, Chicago, Ill.

There is a very slight variation in the angular velocity ratio over an angular displacement or movement of the sprocket equal to the central angle included between the center lines of a pair of adjacent teeth. This variation becomes even smaller as the distance between the center lines of adjacent teeth (the pitch of the sprocket or its chain) is reduced; so that with a fair number of teeth on the smaller sprocket of the pair, together with a small pitch, the variation in the speed ratio is of no practical importance.

Chains may be classified as follows:

(*a*) Hoisting and hauling chains
(*b*) Elevator and conveyor chains
(*c*) Power-transmission chains

The two main types of chains that are used for power transmission are the *Roller Chain* and the *Silent Chain*.

A double roller chain is shown in Fig. 87. Where the power requirement is less, a single roller chain can be used. On the other hand, if the power requirement is too great for a double roller, a triple or quadruple roller chain may be used. Such a chain operates well at speeds as high as 1000 feet per minute.

Since a chain is composed of a great number of small parts, close attention must be paid to its lubrication and cleaning. Where it is exposed to considerable dirt and dust, a casing should be used. Such a casing for the chain affords protection to the employees as well. Periodically the chain should be removed from the sprockets and cleaned in a bath of gasoline or kerosene. After this it should be well drained and its lubricant restored by placing it in a bath of hot grease or oil until the lubricant has had plenty of time in which to penetrate to all bearing surfaces. A single roller chain drive is illustrated in Fig. 88.

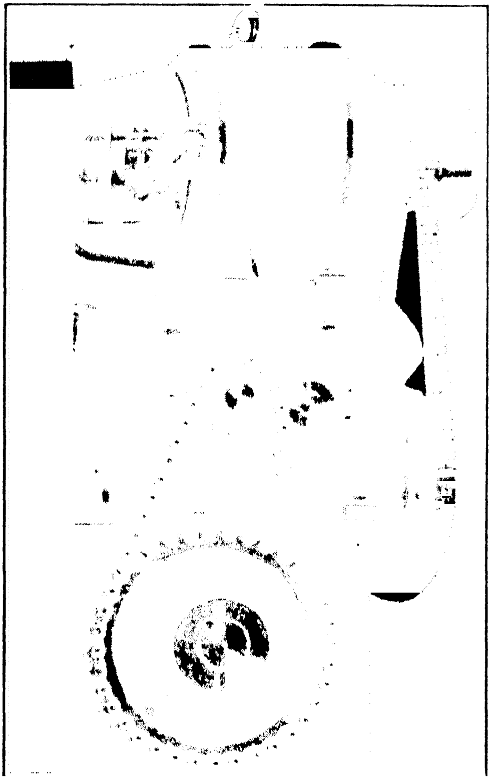


Fig 88 Single Roller Chain Drive
Courtesy of Grant Gear Works, Boston, Mass.

The so-called silent chain of Fig. 89 is an improvement upon the previous type. It gets its name from the fact that it is much less noisy in operation than other types of chains. It can also be used at higher speeds and can transmit a greater horsepower. The Link-Belt Silver-Streak Silent Chain is made in various sizes to transmit power up to 2000 horsepower and over. The Morse Chain Company has silent chains available from the smallest fractional horsepower up to almost any top, and state in their catalog that a 5000 horsepower chain has been made. Silent chains permit of speed reductions of 15 to 1 and of short distances between centers. They usually operate in a casing which contains the oil for automatic lubrication of the drive. A silent chain drive is illustrated in Fig. 90.

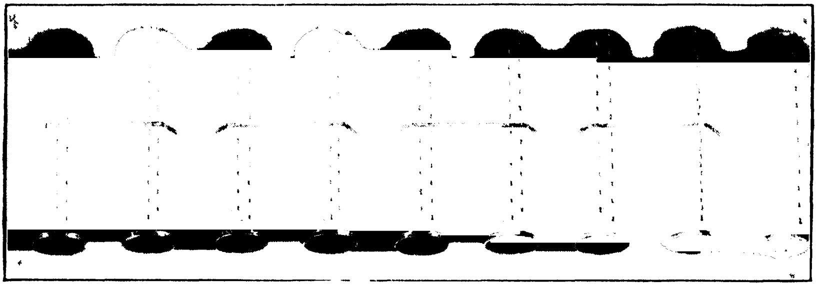


Fig 89 Silent Chain
Courtesy of Link-Belt Company, Chicago, Ill.

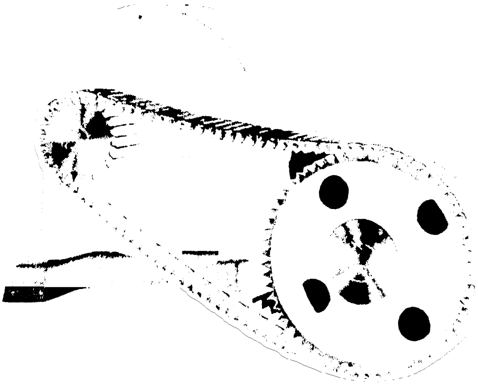


Fig 90 Silent Chain Drive
Courtesy of Link-Belt Company, Chicago, Ill

Example. A reduction in speed from 1000 to 120 revolutions per

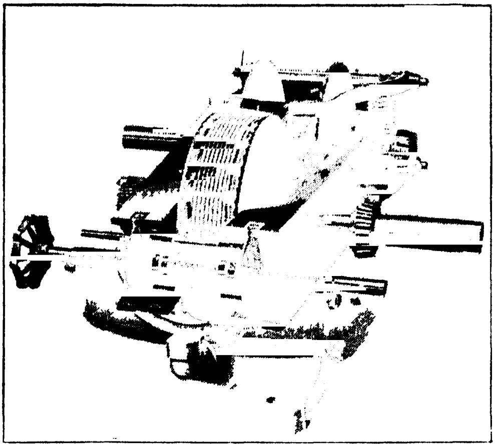


Fig 91(a) Link-Belt P I V Gear

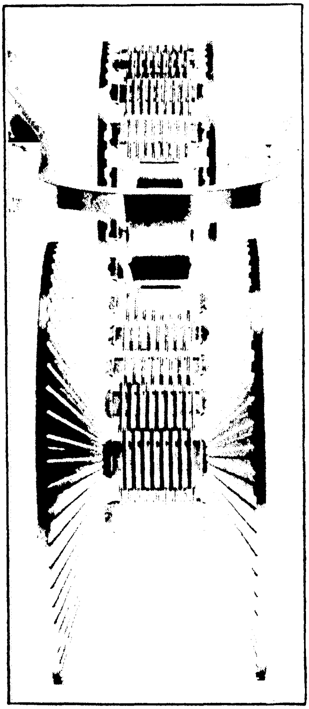


Fig 91(b) Chain on Minimum Diameter (Note the radial teeth on conical wheels)

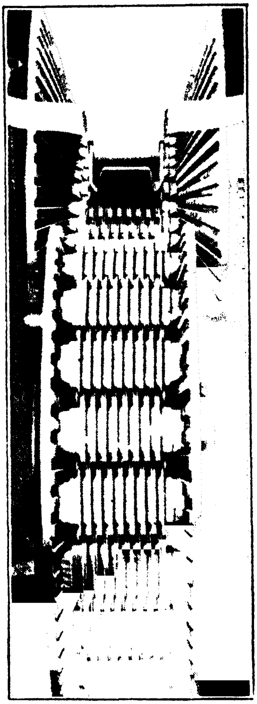


Fig 91(c) Chain on Maximum Diameter

Figs 91(a), 91(b) and 91(c) *Courtesy of Link-Belt Company, Chicago, Ill* minute is desired by the use of a chain drive. If the sprocket of the driver has 18 teeth, how many teeth must there be on the follower?

Solution. Here $N_1 = 1000$ r.p.m., $N_2 = 120$ r.p.m. and $t_1 = 18$ teeth. Applying formula (33)

$$\frac{t_2}{t_1} = \frac{N_1}{N_2}$$

and evaluating therein,

$$\frac{t_2}{18} = \frac{1000}{120}$$

$$t_2 = \frac{1000}{120} \times 18$$

$$t_2 = 150 \text{ teeth.} \quad \textit{Ans.}$$

Variable Speed Transmission Using Chain Drive. The Link-Belt P.I.V. (Positive, Infinitely Variable) Gear illustrated in Figs. 91 (*a*), (*b*), and (*c*) is a variable speed control mechanism which obtains its positive driving through the use of a unique type of chain and obtains its variable speed through the use of two pairs of opposed conical wheels. Each pair of these conical wheels forms a single adjustable unit over which the chain operates in connecting the constant-speed driving or power input shaft to the driven or power output shaft. The position of the chain on these units determines their effective diameters and hence also determines their angular velocity ratio. The chain's position is regulated by the handwheel shown at the lower left of Fig. 91 (*a*). By turning the handwheel, one pair of conical wheels is brought more closely together while the other pair is pulled apart. In this manner the self-adjusting chain is raised between one pair so that it works over a larger effective diameter, while at the same time the chain is lowered between the other pair, causing it to work over a smaller effective diameter. Thus the chain can be given that position on the two pairs of wheels which will enable the constant-speed input shaft to give the desired speed to the output shaft. Hence with this gear speed changes are made easily and smoothly without stopping the drive. It is evident that the smallest speed variations are made possible by this device.

The chain and wheels of a P.I.V. gear differ greatly from the chain and sprockets used in most chain drives. In the P.I.V. drive, the chain is made up of a series of links which are filled with packs of hardened steel laminations or slats. These slats are placed at right angles to the links, project about one-eighth inch to each side of the chain, and are free to slide from side to side with respect to each other. By so doing, they are able to bring themselves into engagement with the radial teeth of the wheels [see Fig. 91 (*b*) and (*c*)] and in this manner provide positive driving.

The P.I.V. gears are contained in compact oil-tight cases and are automatically splash lubricated. They are built in five sizes of from one to fifteen horsepower and give speed change ratios up to a maximum of six to one.

QUESTIONS AND PROBLEMS

1. What is a wrapping-connector mechanism?

2. Classify wrapping connectors.

3. With what kinds of wrapping connectors are sheaves used?

4. What kinds of wrapping connectors are (*a*) positive drives (*b*) friction drives?

5. Define pitch diameter as applied to a flat-belt drive.

6. Give a complete statement of the angular velocity theorem.

7. Is the angular velocity theorem of the wrapping-connector mechanism identical to that of the rigid-link connector mechanism?

8. State the condition for a constant angular velocity ratio between driver and follower in a wrapping-connector mechanism.

9. In a flat-belt drive, pulley A is 9 inches in diameter and makes 800 revolutions per minute. Pulley B, having a diameter of 24 inches, is driven by pulley A. It is required to obtain

(*a*) the r.p.m. of pulley B

(*b*) the speed of the belt in feet per minute

(*c*) the rim velocity of pulley A in feet per minute

(*d*) the rim velocity of pulley B in feet per minute

Ans. (*a*) 300 r.p.m. (*b*) 600π f.p.m. (*c*) 600π f.p.m. (*d*) 600π f.p.m.

10. Two pulleys, 48 inches and 18 inches in diameter respectively, have a distance between their centers of 9 feet.

(*a*) What length of open belt should be used to connect them?

(*b*) What length of crossed belt should be used to connect them?

(*c*) If the pulleys are to rotate in opposite directions should an open or crossed belt be used?

Ans. (*a*) 431.7 in. (*b*) 465.1 in.

11. If the smaller pulley of the preceding problem makes 720 revolutions per minute, what is the speed of the larger pulley? *Ans.* 270 r.p.m.

12. In a roller-chain drive, a 40-tooth sprocket is driven by one having 16 teeth. If the smaller sprocket has an r.p.m. of 400, what is the speed of the larger sprocket? *Ans.* 160 r.p.m.

13. A reduction in speed of 9 to 1 is to be secured between two shafts by a silent-chain drive. If the smaller sprocket is given 18 teeth, how many teeth must there be on the larger sprocket? *Ans.* 162 teeth.

14. A reduction in speed of 3.5 to 1 is to be secured by a belt drive. If the diameter of the larger pulley is 42 inches, what diameter must be used for the smaller pulley? *Ans.* 12 inches.

15. A cone pulley drive employs two equal cone pulleys. The pulleys have three steps whose diameters are 4, 7, and 10 inches respectively. It is required to find the speeds of the driven shaft when the driving shaft is making 250 revolutions per minute. *Ans.* 100 r.p.m., 250 r.p.m., 625 r.p.m.

16. If the line of the connector of a wrapping connector mechanism cuts the line of fixed centers at a point between those centers, what is the directional relation between driver and follower?

17. Where should the force employed to shift a belt be introduced?

18. Why are pulleys crowned?

19. When is it necessary to use an idler or tension pulley?

20. State the law of belting.

21. May the direction of motion of the belt be reversed in a twisted-belt drive in which no guide pulleys are employed?

22. Upon what does the pitch diameter of a V-belt sheave depend?

23. Name five favorable characteristics of the V-belt drive.

24. What is a variable-speed transmission?

25. Name and describe the two systems employed in rope drives.

26. A drill press is driven from an overhead constant-speed shaft whose r.p.m. is 500, by means of a pair of cone pulleys having four steps on each pulley. The largest step on the drive shaft is 9 inches in diameter. Using Burmester's Method, obtain the diameters (correct to the nearest one-eighth inch) of the four steps of each of the cone pulleys if the speeds of the driven cone pulley are 343, 477, 662, and 920 r.p.m., and the distance between the axes of the cone pulleys is 6 feet.

Ans. $D_1 = 9$ in. $d_1 = 4\frac{7}{8}$ in.
$D_2 = 8\frac{1}{8}$ in. $d_2 = 6\frac{1}{8}$ in.
$D_3 = 6\frac{3}{8}$ in. $d_3 = 6\frac{5}{8}$ in.
$D_4 = 5\frac{3}{4}$ in. $d_4 = 8\frac{3}{8}$ in.

Note. A beam compass should be used in this graphical solution in order to permit a large enough drawing scale to effect the desired degree of accuracy

CHAPTER VII

DIRECT-CONTACT MECHANISMS

Introduction. In its most elementary form, a *Direct-Contact Mechanism* consists of three links, which are the driver, the follower, and the frame or fixed link. Motion of the driver is imparted to the follower by the direct contact of a surface of the driver with a surface of the follower. These contacting surfaces are known as the acting surfaces and they may be in contact at a point, along a line, or over a finite surface, so that either higher or lower pairing may be used. (See Chapter I.) The shape or contour of the acting surfaces dictates the relative motion between driver and follower.

The action of the acting surface of the driver upon that of the follower may be one of three types:

1. Pure rolling
2. Pure sliding
3. A combination of rolling and sliding

Pure rolling indicates that a point on the acting surface of the driver contacts no two successive points on the acting surface of the follower. On the other hand, if a point on the acting surface of one of the links contacts all of the successive points on the other link within the extent of the points' path, the action is one of pure sliding. When neither pure rolling nor pure sliding takes place, the action between the contacting links is a mixture of both rolling and sliding. This third type of action is that present between the teeth of mating gears (toothed wheels).

The direct-contact mechanism may be used to connect parallel shafts, intersecting shafts (shafts whose axes intersect), or shafts which are in different planes and hence are neither parallel nor intersecting.

Fig. 92 illustrates a phase of the most general case of a direct-contact pair, in which the action between driver and follower is a combination of rolling and sliding. Here links A and B are used to connect two parallel shafts. Rotation of the driver A causes rotation of the follower B. Since the shafts are parallel, the planes of rotation, being always perpendicular to the axes of rotation, are coincident. This permits us to show the mechanism in Fig. 92 as a projection on a plane taken parallel to the plane of motion, so that the axes of rotation of links A and B and their line (or point) of contact are shown in the figure by the points O, O_1, and P respectively. These will be referred to hereafter as the centers of rotation and the point of contact. The point of contact, P, is in reality a point (or a line) on each link and will be called P_a when using it as a point of A, and P_b when using it as a point of B. Since P_a is a point of the driver A, it rotates about the fixed center O with respect to the frame C, while P_b, a point of the follower B, rotates about the fixed center O_1. (**Note.** In instant center notation, O and O_1 would be called *ac* and *bc* respectively.)

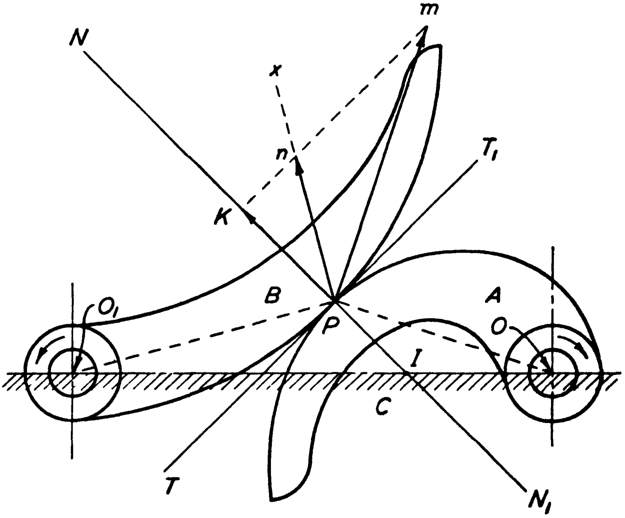


Fig. 92

Common Tangent and Common Normal. Two of the main lines which enter into the discussion of a direct-contact mechanism are the *Common Tangent* and the *Common Normal* to the acting surfaces of the driver and follower at their point of contact, P. In Fig. 92 (see also Fig. 94) TT_1 is the common tangent; NN_1, drawn perpendicular to TT_1 at point P, is the common normal. The common tangent is the line along which any sliding between the acting surfaces of the contact pair will take place. The common normal is the *Line of Action* of the mechanism; that is, it is the line along which the driver must deliver a force or pressure to the follower if motion is to be transmitted without the aid of friction between the driver and follower. The common normal therefore has the same function in the direct-contact mechanism that the line of the connector has in rigid-link and wrapping-connector mechanisms; for the rigid-link connector and the flexible connector are lines of action in their respective modes of transmission of motion.

Relationship of Linear Velocities of Contact Points. A given or assumed angular velocity of the driver of a direct-contact pair dictates or produces a certain linear velocity for the point of contact taken as a point of the driver. This is due to the fact that the linear velocity of a point of a rotating body is equal to the radius of that point, multiplied by the angular velocity of the body. See formula (**16**). Hence, in Fig. 92, multiplying the angular velocity of A by the radius of P_a (which radius is OP) will produce the linear velocity of P_a. The latter is represented in Fig. 92 by the vector Pm which is necessarily drawn perpendicular to the radius OP.

Now the normal component of Pm is obtained by dropping from point m the perpendicular mK to the common normal NN_1. This perpendicular intersects NN_1 at K. Therefore PK is the normal component of Pm. This component indicates the velocity of point P_a in a direction along the common normal.

The point P_b must move along the common normal with the same velocity as point P_a. In other words, the normal component of the linear velocity of P as a point of B must be equal to the normal component of P as a point of A. For if the former were greater than the latter, link B would leave contact with link A; and if it were smaller, link A would tend to occupy the same space as link B. Neither of these conditions is possible, for, in the first place, A is driving B and hence must keep in contact with B; in the second place, A and B are rigid bodies and hence cannot occupy the same space.

Since the normal component of the linear velocity of P_b is known, and since it is known that this linear velocity must be in a direction perpendicular to the radius O_1P of P_b, Px can be drawn perpendicular to O_1P, and Kn can be erected perpendicular to NN_1 and intersecting the indefinite line Px at n. (It should be noted that Kn must coincide with Km.) Therefore Pn is the velocity of P_b to the same velocity scale that is used with Pm, the velocity of P_a.

The foregoing demonstrates that if the linear velocity of the point of contact as a point of the driver is known (or assumed), the linear velocity of the point of contact as a point of the follower may be determined.

Angular Velocity Ratio. Having secured the linear velocity of P_b in Fig. 92, the angular velocity V_{ab} of link B can be obtained by applying formula (17). Thus

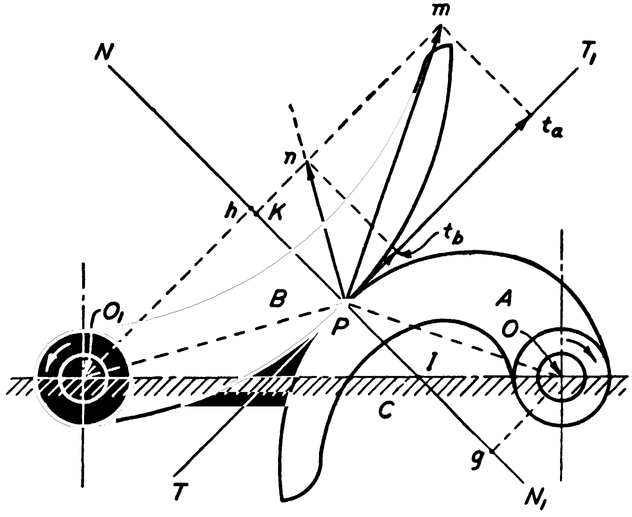


Fig 93

$$V_{ab}=\frac{Pn}{O_1P} \qquad \text{(a)}$$

Applying the same formula in the case of link A,

$$V_{aa}=\frac{Pm}{OP} \qquad \text{(b)}$$

Dividing step (b) by step (a) to obtain the angular velocity ratio of driver to follower,

$$\frac{V_{aa}}{V_{ab}}=\frac{\dfrac{Pm}{OP}}{\dfrac{Pn}{O_1P}} \qquad (34)$$

Therefore it is possible to secure the angular velocity ratio of driver to follower for any phase of a given direct-contact mechanism by simply assuming the linear velocity of the point of contact taken as a point of the driver.

Angular Velocity Theorem. The mechanism of Fig. 92 is reproduced in Fig. 93. In the latter, perpendiculars Og and O_1h are dropped from the fixed centers, O and O_1 respectively, upon the common normal or line of action, NN_1. An inspection of the figure will show that triangle OPg is similar to triangle PmK, and also that triangle O_1Ph is similar to triangle PnK. From triangles OPg and PmK, due to the fact that corresponding sides of similar triangles are proportional, it follows that

$$\frac{Pm}{OP}=\frac{PK}{Og}$$

But $$V_{aa}=\frac{Pm}{OP}$$

therefore $$V_{aa}=\frac{PK}{Og}$$

From triangles O_1Ph and PnK,

$$\frac{Pn}{O_1P}=\frac{PK}{O_1h}$$

But $$V_{ab}=\frac{Pn}{O_1P}$$

therefore $$V_{ab}=\frac{PK}{O_1h}$$

Hence from the foregoing,

$$\frac{V_{aa}}{V_{ab}}=\frac{\frac{PK}{Og}}{\frac{PK}{O_1h}}$$

$$=\frac{PK}{Og}\times\frac{O_1h}{PK}=\frac{O_1h}{Og} \qquad (35)$$

Formula (35) states that in any direct-contact mechanism, at any phase, the angular velocity ratio between driver and follower is inversely as the perpendiculars dropped from the fixed centers upon the common normal.

It may be noted in Fig. 93 that the common normal intersects the line of centers OO_1 at point I; and, in so doing, divides the line of centers internally into the two segments, IO and IO_1. It will also be noted that triangles IOg and IO_1h are similar. In this pair of similar triangles,

$$\frac{IO_1}{IO}=\frac{O_1h}{Og}$$

Hence from formula (35)

$$\frac{V_{aa}}{V_{ab}} = \frac{O_1h}{Og} = \frac{IO_1}{IO} \tag{36}$$

Formula (36) states that in any direct-contact mechanism, at any phase, the angular velocity ratio between driver and follower is inversely as the segments into which the line of fixed centers is cut by the common normal.

The statements of formulas (35) and (36) combine to form the *Angular Velocity Theorem.* Comparison of this with the statements of the angular velocity theorem as given for the other two modes of transmission show all three to be identical if the common normal, the rigid link connector, and the flexible connector are considered as the lines of action for their respective modes.

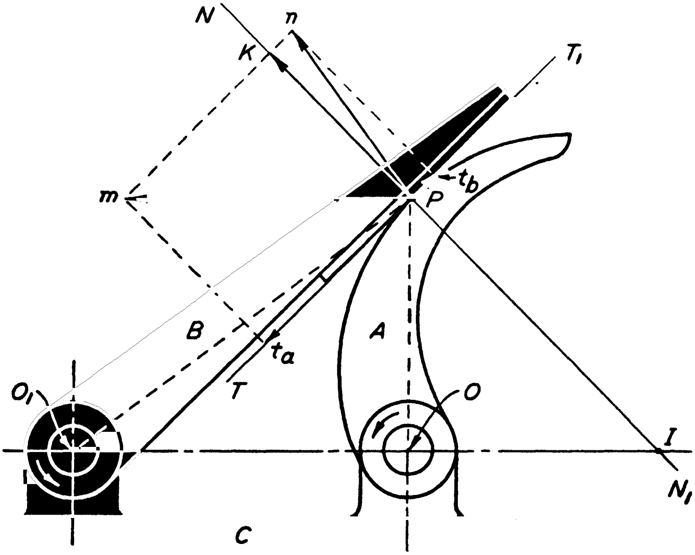


Fig. 94

Constant Angular Velocity Ratio. It is evident from formula (36) that if the ratio $\frac{V_{aa}}{V_{ab}}$ is constant, or the same for all phases throughout the cycle, the ratio $\frac{IO_1}{IO}$ will also be constant. Now O and O_1 are fixed points, and for that reason the ratio $\frac{IO_1}{IO}$ can be constant when, and only when, point I is a fixed point on the line of centers. Hence the condition for a constant angular velocity ratio in a direct-contact mechanism is that the common normal cut the line of fixed centers at the same point at all phases throughout the cycle. It will be shown in the next article that point I is in reality the instant center *ab*. This makes it possible to state the condition for a constant angular velocity ratio in the same manner in which it was stated for the other two modes of transmission of motion. It therefore becomes a general statement for all three modes.

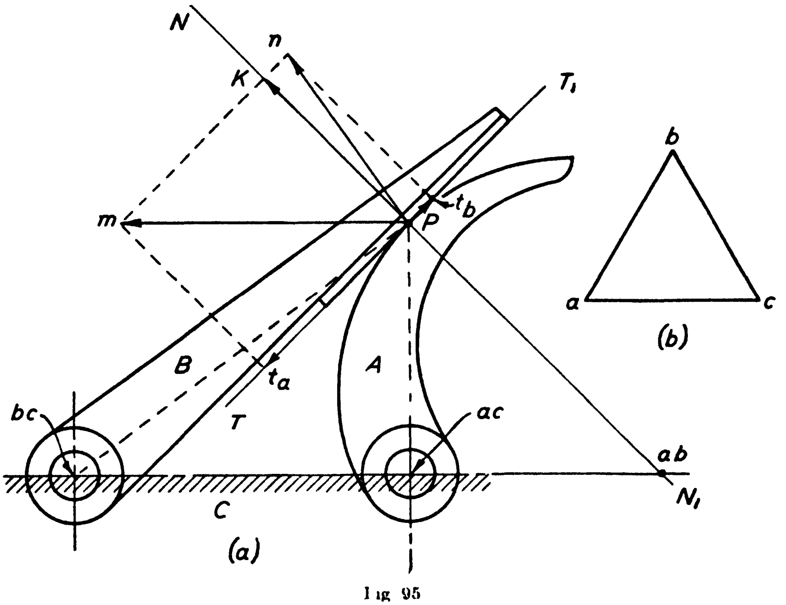


Fig. 95

Instant Centers of a Direct-Contact Mechanism. In the general case of the direct-contact mechanism of Fig. 95 (*a*), links A, B, and C are again the driver, follower, and frame, respectively. Since this mechanism is composed of three links, it is evident that there will be only three instant centers, which, as shown by the instant center polygon of Fig. 95 (*b*) are *ac*, *bc*, and *ab*. The instant center *ac*, of the relative motion of A with respect to C, is a fixed center and can be located by inspection, as can also *bc*, the fixed center of B with respect to C. This leaves only *ab* the instant center of the relative motion of A with respect to B to be determined.

Since Kennedy's theorem states that the three instant centers of any three links of a mechanism must lie on the same straight line, *ab* must lie somewhere on line *ac*-*bc*, the line of fixed centers. It has already been seen that for continuous contact between *A* and *B* the normal component of the linear velocity of *P* as a point of *A* must be equal to the normal component of the linear velocity of *P* as a point of *B*. In other words *PK* is the normal component of both *Pm* and *Pn*. Therefore links *A* and *B* have at a given phase the same motion along the common normal; which is the same as saying that along this line, NN_1, they have no relative motion. Hence the only relative motion of *A* to *B* must be entirely one of sliding in the direction of the tangent TT_1. It was shown in Fig. 31 that when one body is sliding upon another, its instant center with respect to that body is located on a perpendicular to the direction of motion. The instant center, *ab*, must therefore be located on the common normal NN_1 at the intersection of the latter and the line of centers, *ac*-*bc*, as shown in Fig. 95 (*a*). Hence the point called *I* in Figs. 92, 93, and 94 is the instant center *ab*.

Directional Relation. It will be observed in Fig. 92 that rotation of the driver in a clockwise direction causes the follower to rotate in the opposite or counterclockwise direction, and that the common normal (line of action) intersects the line of centers internally or at a point between the fixed centers. On the other hand, it will be observed in Fig. 94 that counterclockwise rotation of the driver *A* produces the same direction of rotation of the follower *B*, and that the common normal intersects the line of centers externally so that the line of centers must be extended to provide the point of intersection *I*. It follows then that if the common normal intersects the line of centers internally, driver and follower rotate in opposite directions, while if the common normal intersects the line of centers externally, driver and follower rotate in the same direction. A comparison of this statement with those statements made for directional relation in the other two modes of transmission of motion will show that directional relation is established in the same manner in all modes.

Rate of Sliding between Driver and Follower. As stated previously in this chapter, the sliding action occurring between the driver and follower of a direct-contact pair is along their common tangent. Hence the tangential components of the linear velocities of the points of contact with respect to the fixed link, or frame, determine the rate of relative sliding between the driver and follower.

Referring to Fig. 93, P_a, the point of contact taken as a point of A, has a linear velocity relative to the frame C, which is equal to (or represented by) Pm. The tangential component of Pm is Pt_a. It is found by dropping a perpendicular from m to TT_1, intersecting the latter at t_a. This tangential component represents the velocity at which P_a is traveling relative to C along TT_1 at the given phase. Now Pn is the linear velocity of P_b (P taken as a point of B) with respect to C. Projecting n on TT_1 at t_b gives the tangential component, Pt_b. The latter shows the velocity at which P_b is traveling relative to C along TT_1. It will be noted that Pt_a and Pt_b are in the same direction along TT_1. Therefore points P_a and P_b are traveling in the same direction along that line.

Although they are traveling in the same direction, their speeds are different; hence one link must slip or slide on the other. The magnitude or rate of this sliding of A with respect to B is obtained by subtracting the smaller component from the larger. Therefore the rate of sliding in Fig. 93 becomes

$$Pt_a - Pt_b = t_b t_a$$

In some instances, such as that shown in Fig. 94, the tangential components, Pt_a and Pt_b, are oppositely directed along the common tangent. In such a case, the rate of sliding is equal to the sum of the absolute values of Pt_a and Pt_b. The rate of sliding in Fig. 94 becomes

$$Pt_a + Pt_b = t_a t_b$$

Note. When two vectors are oppositely directed along the same line, one is considered opposite in algebraic sign to the other. If algebraic signs are used for the tangential components, the two statements just given can be combined as follows: the rate of sliding between driver and follower in a direct-contact mechanism is equal to the algebraic difference of the tangential components Pt_a and Pt_b.

Pure Rolling. From the previous discussion of the rate of sliding, it is evident that if the velocities of the points of contact at a given phase had equal tangential components (same direction and of equal magnitude), the rate of sliding would equal zero or, in other words, no sliding would exist. Hence any action between the driver and follower would be one of pure rolling. Since in any direct-contact mechanism the normal components of the velocities of the contact points are equal, the points of contact of a direct-contact pair having pure rolling have identical velocities.

The previous statement is brought out graphically in Fig. 96 (see also Fig. 104) in which, through point P, representing the point of contact of a direct-contact pair having pure rolling, line TT_1 is drawn as the common tangent and NN_1, perpendicular to TT_1 at point P, is drawn representing the common normal. Assume PK as the normal component and Pt_a as the tangential component of the velocity of P considered as a point of the driver, and lay these components off from point P on lines NN_1 and TT_1 respectively, as shown. Complete the parallelogram of velocities by drawing lines through K and t_a parallel to TT_1 and NN_1 respectively. The diagonal Pm of the parallelogram thus formed is the resultant of the two component velocities PK and Pt_a, and hence is the velocity of P taken as a point of the driver.

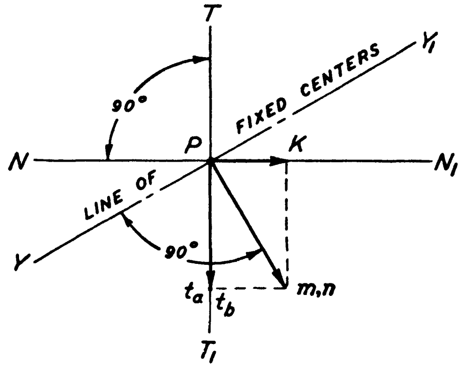


Fig. 96

Now consider P as a point of the follower. The normal component of its velocity must be equal to PK, and the tangential component Pt_b must be equal to Pt_a, since the action is one of pure rolling. These components coincide with those previously drawn; therefore they produce the same resultant; and Pn, the velocity of P as a point of the follower, is coincident with and equal to Pm. Hence in a direct-contact mechanism having pure rolling, the linear velocities of the contact points are equal.

A point of a rotating body has a direction of motion along a tangent to its circular path. The center of rotation of the point must therefore be located on a radius drawn through the point perpendicular to its direction of motion. Hence, for example, in Fig. 96 the center of rotation of P taken as a point of the driver is somewhere on line YY_1 which is drawn through P perpendicular to Pm. Likewise, the center of rotation of P taken as a point of the follower is somewhere on a line which is drawn through P perpendicular to Pn. Since Pm and Pn are coincident, only one perpendicular, YY_1, can be drawn to them at point P. Therefore the fixed centers of both driver and follower lie on line YY_1, which becomes the line of centers and contains the point of contact, P. Hence whenever, in a direct-contact pair, the point of contact lies on the line of centers, the action between driver and follower is one of pure rolling; and, as motion continues, the pure rolling will persist only as long as the point of contact continues to be so located.

Instant Center of a Rolling Direct-Contact Pair. It has been pointed out that the instant center of driver and follower in any direct-contact mechanism is at the intersection of the common normal and the line of centers. As shown in Figs. 96, 97, and 104, where pure rolling is involved, this intersection is at the point of contact P. Therefore in direct-contact mechanisms which have pure rolling, the instant center of the relative motion of driver to follower is at the point of contact.

The preceding statement is in full agreement with the definition of an instant center; for P as a point of the driver has the velocity Pm and as a point of the follower has the velocity Pn, which is identical to Pm. Having the same velocities with respect to the fixed link (the same absolute velocities), the points of contact have no relative motion with respect to each other and hence by definition are at the instant center of their respective links.

Angular Velocity Ratio in a Rolling Direct-Contact Pair. It has been proved that the angular velocity ratio between driver and follower in any direct-contact mechanism is inversely as the segments into which the line of centers is cut by the common normal. In any direct-contact mechanism, the common normal passes through the point of contact, and in a direct-contact mechanism having pure rolling, as in Fig. 97, the contact point is on the line of centers. Therefore the segments into which the common normal NN_1 of Fig. 97 cuts the line of centers, *ac-bc*, are *ac-ab* and *bc-ab*. But *ac-ab* is the contact radius of link A and *bc-ab* is the contact radius of link B. Therefore in a direct-contact mechanism having pure rolling (and only in those having pure rolling) the angular velocity ratio of driver to follower is inversely as the contact radii of the given phase. Writing this as a formula,

$$\frac{V_{aa}}{V_{ab}}=\frac{bc\text{-}ab}{ac\text{-}ab} \tag{37}$$

In Fig. 97, drop perpendiculars *ac-r* and *bc-s* from the fixed centers *ac* and *bc*, to the common tangent TT_1. By so doing, two similar triangles, *ac-ab-r* and *bc-ab-s*, are formed. In these similar triangles,

$$\frac{bc\text{-}ab}{ac\text{-}ab}=\frac{bc\text{-}s}{ac\text{-}r}$$

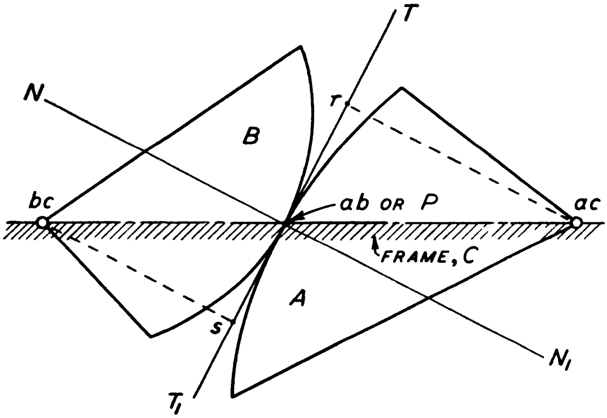


Fig. 97

Since things equal to the same thing are equal to each other, from this equation and formula (37),

$$\frac{V_{aa}}{V_{ab}}=\frac{bc\text{-}s}{ac\text{-}r} \tag{38}$$

Formula (38) states that in a direct-contact mechanism having pure rolling, the angular velocity ratio of driver to follower is inversely as the perpendiculars which are dropped from the fixed centers to the common tangent.

Profiles or Outlines of Pure Rolling Members. To secure pure rolling continuously between a direct-contact pair, there are many types of curves that can be used as the outlines of the acting surfaces. Some of these are pure geometric shapes such as circles or circular arcs, ellipses, and logarithmic spirals. Under certain imposed conditions, each type, when used as the outline of a pair of contacting surfaces, will cause the point of contact to remain on the line of centers and will produce successive angular velocity ratios which are peculiar to itself. Other curves, more general in form, can be derived to roll with each other and give required angular velocity ratios at successive phases. It is also possible to construct a curve which will have pure rolling with one that is given or known.

Pure Rolling and Constant Angular-Velocity Ratio. For a direct-contact mechanism to have a combination of pure rolling and constant angular-velocity ratio, the point of contact must always lie on the line of centers, and the ratio of the contact radii must be constant. Since the sum of the contact radii must be equal to the distance between the fixed centers of the contact pair, each contact radius must be constant in length if the ratio of the contact radii is to be constant. The only outlines that a direct-contact pair may have to meet these conditions are circles which rotate about their geometric centers as fixed centers. Such rolling circles can be the transverse sections of contact pairs which connect shafts whose axes are variously located with respect to each other.

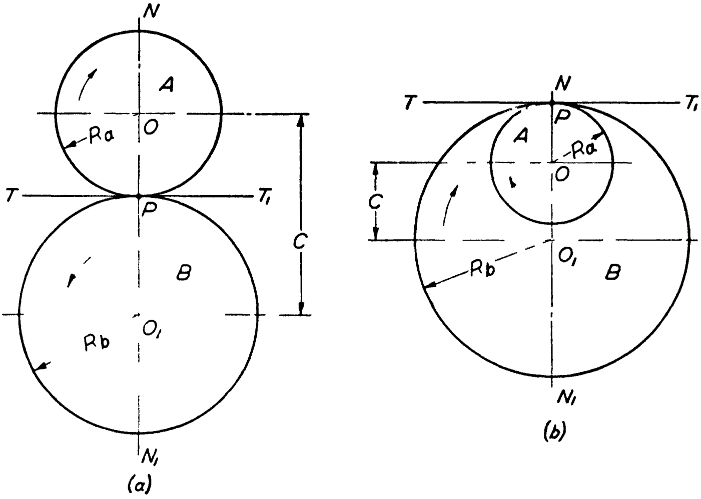


Fig 98 Rolling Circles

Two pairs of right circular cylinders are shown in Fig. 98 (*a*) and (*b*). Here the cylinders are projected on planes which are parallel to the planes of rotation so that the circular projections represent true transverse sections of the cylinders. It will be noted in Fig. 98 (*a*) that the circles are tangent externally at the contact point P and that clockwise rotation of A produces counterclockwise rotation of B, while in Fig. 98 (*b*), the circles are tangent internally at the contact point P, and clockwise rotation of A produces the clockwise rotation of B. So with external tangency the circles (or cylinders) rotate oppositely, while with internal tangency they rotate in the same direction. Applying formula (11) with

R_a = the radius of A
R_b = the radius of B
N_a = the r.p.m. of A
N_b = the r.p.m. of B

the linear velocity of P_a in either Fig. 98 (*a*) or 98 (*b*) is $2\pi R_a N_a$ and the linear velocity of P_b is $2\pi R_b N_b$. Since, in pure rolling, these velocities are equal, by equating one to the other,

$$2\pi R_a N_a = 2\pi R_b N_b$$

Dividing both members of this equation by 2π,

$$R_a N_a = R_b N_b$$

Dividing both members by $R_a N_b$,

$$\frac{N_a}{N_b} = \frac{R_b}{R_a} = \text{a constant} \tag{39}$$

Multiplying both numerator and denominator of the second member of formula (39) by 2,

$$\frac{N_a}{N_b} = \frac{2R_b}{2R_a} = \frac{D_b}{D_a} = \text{a constant} \tag{40}$$

Formulas (39) and (40) state that the r.p.m. ratio of a pair of rolling circles (or cylinders) is constant, and equal to the inverse ratio of their radii or diameters.

Since the ratio of the angular velocities of driver and follower in any mode of transmission of motion is equal to their r.p.m. ratio, from formulas (39) and (40),

$$\frac{V_{aa}}{V_{ab}} = \frac{R_b}{R_a} = \frac{D_b}{D_a} = \text{a constant} \tag{41}$$

Example. In Fig. 98 (*a*), what is the r.p.m. of B if the diameter of A is 10 inches, the diameter of B is 27 inches, and the rotative speed of A is 135 r.p.m.?

Solution. Here $D_a = 10$ in., $D_b = 27$ in., and $N_a = 135$ r.p.m. Applying formula (40), since in this example diameters are given,

$$\frac{N_a}{N_b}=\frac{D_b}{D_a}$$

Since the reciprocals of equal ratios are equal,

$$\frac{N_b}{N_a}=\frac{D_a}{D_b}$$

Evaluating,

$$\frac{N_b}{135}=\frac{10}{27}$$

Multiplying both members of the above by 135,

$$N_b=\frac{10}{27}\times 135=50 \text{ r.p.m.} \quad \textit{Ans.}$$

Example. In the preceding example, it is required to find:

(a) V_{La}, the linear velocity of the point of contact, P_a, in f.p.m.

(b) V_{Lb}, the linear velocity of the point of contact, P_b, in f.p.m.

(c) V_{aa}, the angular velocity of A, in radians per second.

(d) V_{ab}, the angular velocity of B, in radians per second.

(e) the value of the angular velocity ratio, $\frac{V_{aa}}{V_{ab}}$.

(f) the value of the r.p.m. ratio, $\frac{N_a}{N_b}$.

Solution. (a) Here $D_a=\frac{10}{12}=\frac{5}{6}$ ft., and $N_a=135$ r.p.m. Applying formula (**11**)

$$V_{La}=\pi D_a N_a$$

and evaluating therein,

$$V_{La}=\pi\times\frac{5}{6}\times 135=353.43 \text{ f.p.m.} \quad \textit{Ans.}$$

(b) Here $D_b=\frac{27}{12}=\frac{9}{4}$ ft. and $N_b=50$ r.p.m. Applying formula (**11**)

$$V_{Lb}=\pi D_b N_b$$

and evaluating therein,

$$V_{Lb}=\pi\times\frac{9}{4}\times 50=353.43 \text{ f.p.m.} \quad \textit{Ans.}$$

(c) Applying formula (**14**)

$$V_{aa}=2\pi N_a$$

$$=2\times\pi\times 135=848.23 \text{ radians per minute}$$

or $$V_{aa}=\frac{848.23}{60}=14.14 \text{ radians per second.} \quad \textit{Ans.}$$

(*d*) Applying formula (14)

$$V_{ab} = 2\pi N_b$$
$$= 2 \times \pi \times 50 = 314.16 \text{ radians per minute}$$

or $$V_{ab} = \frac{314.16}{60} = 5.24 \text{ radians per second.} \quad \textit{Ans.}$$

(*e*) $$\frac{V_{aa}}{V_{ab}} = \frac{14.14}{5.24} = 2.7 \quad \textit{Ans.}$$

(*f*) $$\frac{N_a}{N_b} = \frac{135}{50} = 2.7 \quad \textit{Ans.}$$

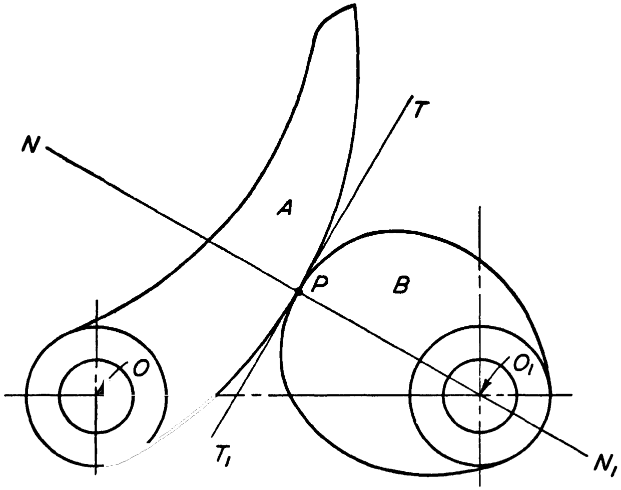


Fig. 99

Positive Driving. It has been stated that in a direct-contact mechanism a force or pressure must be delivered by the driver to the follower along their common normal whenever motion is to be transmitted without the aid of friction. Such a transmission of motion would be the result of what is known as *Positive Driving,* and it is due to the character of the outlines or profiles of the contact surfaces. (See Figs. 93 and 94 which are general cases and give positive driving.) While such a normal force must be present if positive driving is to take place, it does not follow that such a normal force is assurance of positive driving.

In Fig. 99 it is evident that if the driver A tends to turn in a clockwise direction, it will press on B and hence deliver a force which will act along the common normal NN_1. Since the latter passes through the fixed center of B and is therefore coincident with the contact radius of B, a force along it will have no tendency to turn the follower about its axis of rotation and will only set up a pressure between the shaft (to which B is keyed) and the bearings that support the shaft. Since B is a rigid body, A will be locked in position by it. Hence positive driving cannot take place if the common normal passes through the fixed center of the follower.

Now let us consider Fig. 100 which is composed of the same links as Fig. 99 but in which the links are interchanged so that the driver,

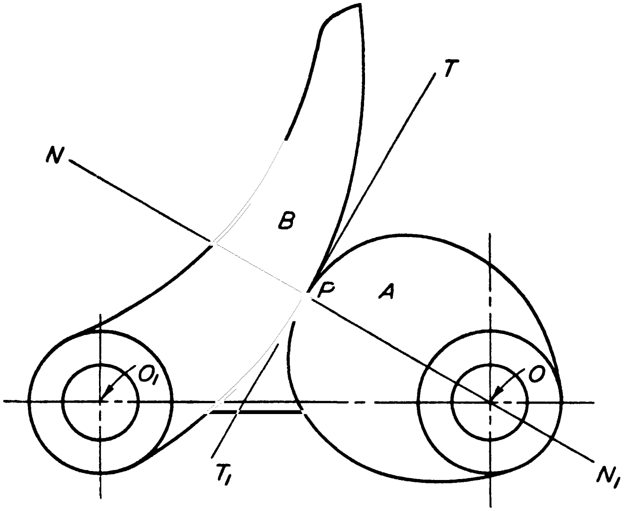


Fig. 100

A, of Fig. 99 has become the follower, B, of Fig. 100; and the follower, B, of Fig. 99 has become the driver, A, of Fig. 100. It should be noted in this case that the velocity of P_a, acting of course in a direction perpendicular to the radius OP, will coincide with the common tangent TT_1 and hence can have no normal component. Evidently, then, there can be no positive driving of B by A. However an inspection of the figure shows that A will not be locked in position by the follower B, hence the driver A can rotate about its fixed center. This absence of positive driving is due to the fact that the contact radius of A is coincident with the common normal, so that the latter passes through the fixed center of the driver. Hence positive driving cannot take place if the common normal passes through the fixed center of driver.

Now refer to either Fig. 98 (*a*) or Fig. 98 (*b*), where the common normal passes through not only the fixed center of the driver but also through the fixed center of the follower. Inspection of the figure shows that the velocity of the contact point is along the common tangent, so that there is no normal component thereof. It is also evident that if the contact surfaces of *A* and *B* were perfectly smooth, so as to be frictionless, the driver could continue to rotate about its axis with no effect whatever on the follower. Therefore, positive driving cannot occur when the common normal passes through both fixed centers and hence is coincident with the line of centers.

From previous statements of this article, it can be stated that it is possible to obtain positive driving in a direct-contact mechanism when, and only when, the common normal does not pass through the center of rotation of either the driver or the follower.

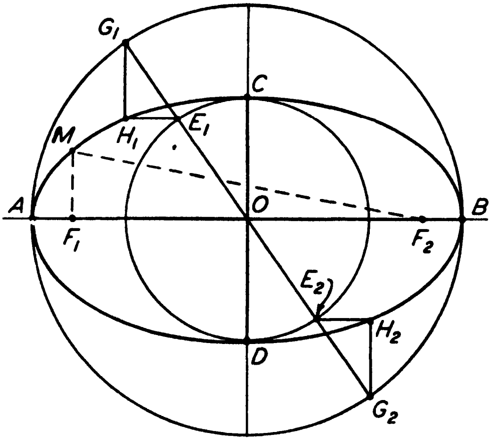


Fig. 101

Definition and Construction of an Ellipse. An ellipse is shown in Fig. 101. It is a plain geometric figure having two axes, *AB* and *CD*, which are called the major and minor axes respectively. These axes bisect each other at right angles. Two points, F_1 and F_2, called the foci, are located on the major axis. The main characteristic of the ellipse is that the sum of the distances from any point on its circumference to the two foci is equal to the length of the major axis. Thus in Fig. 101, $MF_1+MF_2=AB$.

There are several methods of constructing an ellipse, one of which is shown in Fig. 101. Here the given axes, *AB* and *CD*, are laid off so that they bisect each other at right angles at point *O*, which is known as the center of the ellipse. With *O* as a center and a radius *OC*, equal to one-half of the minor axis, describe a circle. Likewise with *O* as a center and a radius *OA*, equal to one-half of the major axis, describe another circle. Draw a diameter, G_1G_2, of the outer circle, which diameter intersects the smaller circle at E_1 and E_2. Through G_1 draw a line parallel to the minor axis, and through E_1 draw a line parallel to the major axis. These lines intersect at H_1, which is a point on (the circumference of) the ellipse. In a similar manner G_2H_2 and E_2H_2, drawn parallel to the minor and major axes respectively, intersect at point H_2, which is another point on the ellipse. By drawing several diameters of the outer circle and proceeding with each as shown with G_1G_2, other points on the ellipse can be obtained. These points are then connected by a smooth curve to form the ellipse.

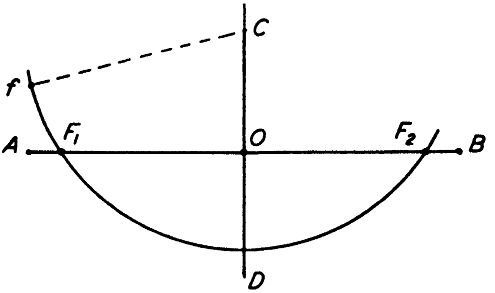


Fig 102

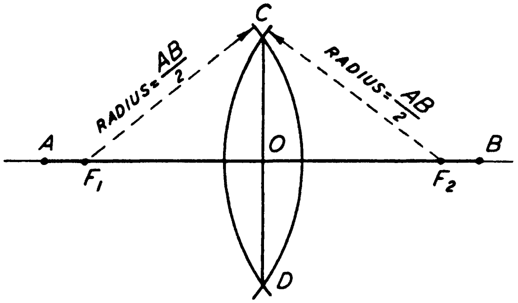


Fig. 103

Fig. 102 shows the method by which the foci of an ellipse can be located when the major and minor axes are known. The first step is to lay off the two axes, *AB* and *CD*, so that they intersect each other at right angles. Then with *C* (the end of the minor axis) as a center, and with *Cf* (equal to *OA*, one-half of the major axis) as a radius, describe an arc which intersects the major axis at points F_1 and F_2. These points are the foci of the ellipse.

Sometimes only the major axis and the location of the foci thereon are given for constructing an ellipse. It is then necessary to obtain the minor axis of the ellipse. Referring to Fig. 103, lay off the given major axis, *AB*, and locate thereon the given foci, F_1 and F_2. With F_1 and F_2 as centers and one-half of the major axis as a radius, describe arcs intersecting at points *C* and *D*. The latter are the ends of the minor axis which can now be drawn, and which will intersect *AB* at right angles at point *O*, the center of the ellipse.

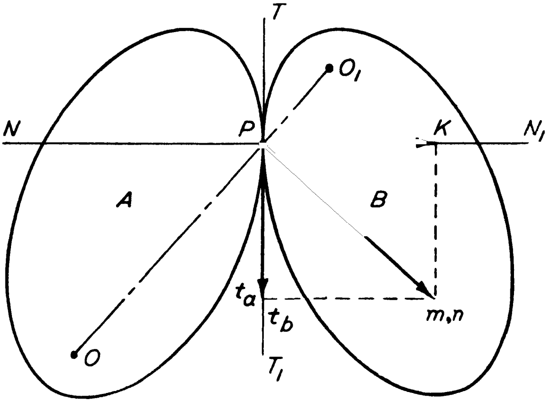


Fig. 104. Rolling Ellipses

Rolling Ellipses. Two ellipses used as a direct-contact pair will give pure rolling under the following conditions:

1. The ellipses must be equal; that is, their major, as well as their minor, axes must be of the same length.

2. The ellipses must rotate about one of their foci as a fixed center.

3. The distance between the fixed centers must be equal to the length of the major axis.

Two ellipses that satisfy the above conditions are shown at different phases in Figs. 104 and 105. In each of these figures, *A* may be considered as the driver and B as the follower. A and B rotate about their respective foci, O and O_1, as fixed centers. These links A and B have pure rolling at the given phases because their point of contact is on the line of centers. They will continue to have pure rolling at all succeeding phases because by satisfying the above conditions, the point of contact will continue to lie on the line of centers throughout the cycle.

In Fig. 104 the common normal does not pass through the fixed center of either link; therefore, at the phase shown, the driving is positive. In fact the driving is positive for a half turn, at the end of which the driver and follower take the position shown by Fig. 105. In the latter position or phase, it will be noted that the common normal passes through both fixed centers and hence positive driving ceases to exist. The driver can continue to rotate, but the follower will cease to be affected by it. It is for this reason that, when rolling ellipses must make complete turns about their axes, their contact surfaces are provided with interlocking members, called teeth, in order to secure positive driving.

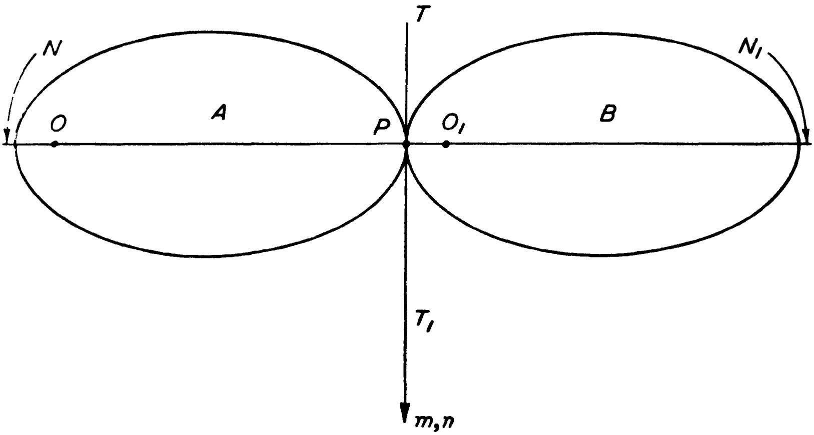


Fig 105 Rolling Ellipses

It has been proved that, in pure rolling, the angular velocity ratio of driver to follower is inversely as the contact radii, or

$$\frac{V_{aa}}{V_{ab}}=\frac{O_1P}{OP}$$

Solving for the angular velocity V_{ab} of the follower,

$$V_{ab}=\frac{OP}{O_1P}\times V_{aa}$$

If V_{aa}, the angular velocity of the driver, is constant, the preceding equation will yield V_{ab}, the maximum angular velocity of the follower, when the contact radius OP of the driver is a maximum and the contact radius O_1P of the follower is a minimum. This occurs in the phase shown in Fig. 105. At a phase 180 degrees removed from that of Fig. 105, the ellipses will again have their major axes in the same straight line, and the angular velocity of the follower will be a minimum. Due to the fact that the ellipses are equal, the minimum angular velocity of the follower is the reciprocal of the maximum. For all the other phases, the angular velocity of the follower will vary gradually from its maximum to its minimum value and then from its minimum to its maximum value during a complete turn.

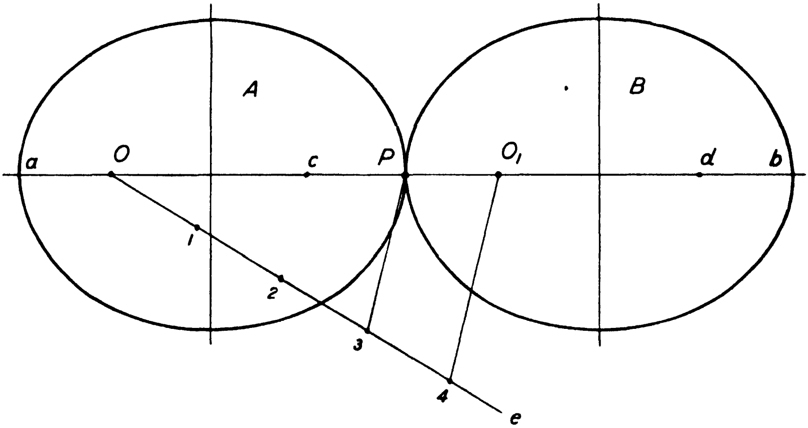


Fig 106

Construction of Ellipses for a Desired Velocity Ratio. It is possible to construct a pair of ellipses to secure a given maximum, and hence minimum, angular velocity of the follower. Let it be assumed that O and O_1 of Fig. 106 are the fixed centers of the driver A and the follower B, and that $\frac{V_{aa}}{V_{ab}}=\frac{1}{3}$. From this ratio is obtained $V_{ab}=3V_{aa}$. Since the ratio is $\frac{1}{3}$, it is first necessary to divide line OO_1 into 3+1, or 4, equal divisions. Draw the indefinite line Oe, making any angle with

OO_1; and upon Oe, from point O lay off points *1*, *2*, *3*, and *4* equidistant from each other. Draw a line from *4* to O_1. Now draw a line, through *3*, parallel to O_14 and intersecting OO_1 at P. O_1P equals OO_1 divided by 4 and O_1P is therefore one-third of OP, so that O_1P becomes the contact radius of B, and OP becomes the contact radius of A with P, the point of contact.

Lay off Pa and Pb equal to the distance between centers OO_1. Then Pa and Pb are the major axes of A and B respectively. Next locate the other two foci, c and d. They are at a distance equal to O_1P from points P and b respectively. Now with the major axis of each ellipse and the position of its foci known, the minor axis is obtained by the method of Fig. 103. With their major and minor axes known, the ellipses may be constructed by the method of Fig. 101.

Friction Drives. When a direct-contact mechanism has pure rolling but does not have positive driving, its only possibility of transmitting motion is through the use of a frictional resistance that may be set up between the contact surfaces. Whenever friction is employed, some slipping is liable to occur. During this slipping, the acting surfaces of course must remain in contact with each other. To do this, the radii of each member of the contact pair must be constant. Therefore, whenever friction is employed, the driver and follower must have circular transverse sections. Since, in pure rolling, the angular velocity ratio of driver to follower is inversely as the contact radii, friction drives with their circular transverse sections will have, theoretically, a constant angular velocity ratio. They are designed to give such a constant ratio, but it is recognized that with any slippage the velocity ratio will be variable and uncertain; hence their use is with those machines in which this slight variation in speed ratio is of no consequence. Of course where no such variation may be permitted, a positive drive must be used.

Cylindrical or Spur Friction Wheels. Parallel shafts may be connected by a friction drive which is composed of a pair of right circular cylinders as shown in Figs. 107 and 108. Such cylinders are known as *Cylindrical* or *Spur Friction Wheels.* With them, transmission of motion is dependent on a frictional resistance which is set up along their common contact element, the line of contact, which in both figures is shown by the projection P of the front elevation. Assuming that no slipping occurs, this type of drive gives pure rolling and a constant angular velocity ratio. Like all friction direct-contact drives, its dependence on friction is due of course to the fact that the common normal is so located that positive driving is not possible.

Case I. In this case the cylinders are in contact or are tangent externally as in Fig. 107, and the parallel shafts connected by them rotate in opposite directions. It is required to obtain the radii or diameters of that pair of rolling cylinders to be used to connect a pair of such shafts when the distance between centers and the angular velocity or r.p.m. ratio are known. Referring to Fig. 107, let

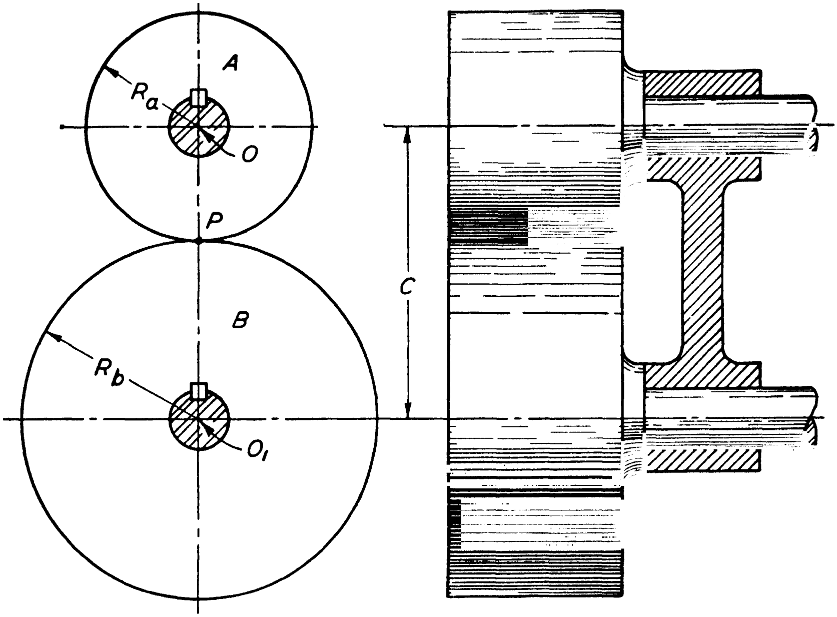


Fig. 107 Rolling Cylinders External Contact

R_a = radius of driver A

R_b = radius of follower B

D_a = diameter of driver A

D_b = diameter of follower B

N_a = r.p.m. of driver A

N_b = r.p.m. of follower B

C = distance between centers

It is evident from the figure that

$$R_a + R_b = C \tag{a}$$

From formula (39),

$$\frac{R_b}{R_a}=\frac{N_a}{N_b}$$

Multiplying both members of the above equation by R_a,

$$R_b=\frac{N_a}{N_b}\times R_a \qquad \text{(b)}$$

These equations, (*a*) and (*b*), can now be used as a pair of simultaneous equations in solving for R_a and R_b.

Case II. In the case of Fig. 108, the cylinders are in contact or are tangent internally and the parallel shafts connected by them rotate in the same direction. It is required to obtain the radii or diameters of the rolling cylinders to be used to connect the shafts when the distance between centers and the angular velocity or r.p.m. ratio are known. The notation in use is the same as that for *Case I.*

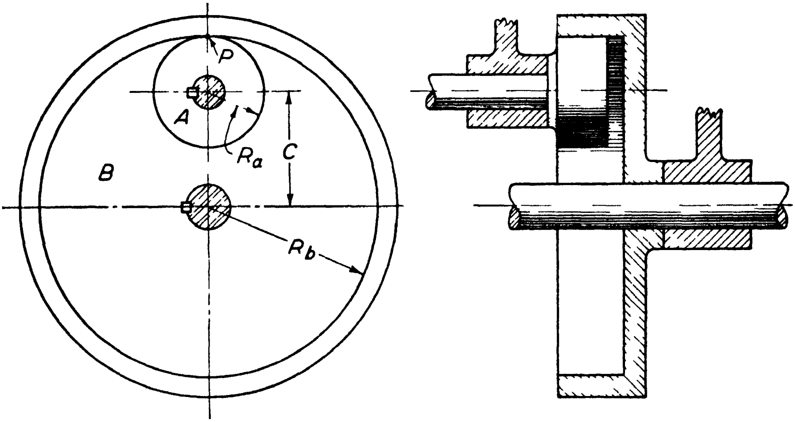


Fig. 108 Rolling Cylinders Internal Contact

It is evident from the figure that

$$R_b-R_a=C \qquad \text{(a)}$$

Note. In this equation, R_b should be larger in value than R_a. The larger cylinder will have the smaller r.p.m. Why? Hence in this case always give the name *B* to the cylinder with the smaller r.p.m.

From formula (39),

$$\frac{R_b}{R_a}=\frac{N_a}{N_b}$$

Solving the above equation for R_b,

$$R_b=\frac{N_a}{N_b}\times R_a \qquad \text{(b)}$$

These equations (*a*) and (*b*) can now be used in a manner similar to those of the preceding case in solving for R_a and R_b.

Example 1. Two parallel shafts rotating in opposite directions are to be connected by a pair of rolling cylinders. It is required to find the diameters of the cylinders to be used if $N_a=150$ r.p.m.; $N_b=90$ r.p.m.; $C=12$ in.

Solution. Step 1. Since the shafts rotate in opposite directions, *Case I* applies.

Evaluating in step (*a*) of *Case I.*

$$R_a+R_b=12$$

Step 2. Applying step (*b*) of *Case I* (or formula **39**)

$$R_b=\frac{N_a}{N_b}\times R_a$$

and evaluating therein

$$R_b=\frac{150}{90}\times R_a=\frac{5R_a}{3}$$

Step 3. Substituting, in step *1*, the value of R_b obtained in step *2*,

$$R_a+\frac{5R_a}{3}=12$$

Multiplying both members of the equation by 3 to clear the equation of fractions

$$3R_a+5R_a=36$$

Collecting terms,

$$8R_a=36$$

$$R_a=\frac{36}{8}=4.5$$

Therefore $\quad D_a=2R_a=2\times4.5=9$ in. *Ans.*

Step 4. Substituting, in step *2*, the value of R_a obtained in step *3*,

$$R_b=\frac{5\times4.5}{3}=\frac{22.5}{3}=7.5 \text{ in.}$$

Therefore $\quad D_b=2R_b=2\times7.5=15$ in. *Ans.*

Step 5. To check the results, substitute in step *1* the values obtained for R_a and R_b

$$R_a+R_b=12$$

$$4.5+7.5=12 \text{ in.} \qquad \text{to check}$$

Example 2. Two parallel shafts rotating in the same direction are to be connected by a pair of rolling cylinders. It is required to find the diameters of the cylinders to be used if $N_a=140$ r.p.m.; $N_b=50$ r.p.m.; $C=6$ in.

Solution. Step 1. Since the shafts rotate in the same direction, *Case II* applies.

Evaluating in step (*a*) of *Case II*,

$$R_b - R_a = 6$$

Step 2. Applying step (*b*) of *Case II*

$$R_b = \frac{N_a}{N_b} \times R_a$$

and evaluating therein

$$R_b = \frac{140}{50} \times R_a = \frac{14R_a}{5}$$

Step 3. Substituting, in step 1, the value of R_b obtained in step *2*

$$\frac{14R_a}{5} - R_a = 6$$

Multiplying both members of the equation by 5 to clear the equation of fractions

$$14R_a - 5R_a = 30$$

Collecting terms

$$9R_a = 30$$

$$R_a = \frac{30}{9} = 3\tfrac{1}{3} \text{ in.}$$

Therefore $\quad D_a = 2R_a = 2 \times 3\tfrac{1}{3} = 6\tfrac{2}{3}$ in. *Ans.*

Step 4. Substituting, in step *2*, the value of R_a obtained in step *3*,

$$R_b = \frac{14R_a}{5} = \frac{14 \times 3\frac{1}{3}}{5} = \frac{\frac{140}{3}}{5}$$

$$= \frac{140}{3} \times \frac{1}{5} = \frac{140}{15} = 9\tfrac{1}{3} \text{ in.}$$

Therefore $\quad D_b = 2R_b = 2 \times 9\tfrac{1}{3} = 18\tfrac{2}{3}$ in. *Ans.*

Step 5. To check the results, substitute, in step *1*, the values obtained for R_a and R_b.

$$R_b - R_a = 6$$

$$9\tfrac{1}{3} - 3\tfrac{1}{3} = 6 \text{ in.} \qquad \text{to check}$$

Rolling Cones or Bevel Frictions. Two shafts whose axes intersect at some angle, θ (theta) as in Fig. 109, can be connected by a pair of right circular cones or their frusta. Such rolling conical frusta, which are always used in practice instead of the entire cones, are known as *Bevel Friction Wheels.* They can be designed for any angular velocity ratio and any angle, θ, between the shafts. While from a theoretical standpoint they maintain a constant angular velocity ratio together with pure rolling, their driving action is not positive, but depends wholly on the frictional resistance which is set up along their common contact element, *Kg*. Hence, while transmitting motion, some slippage is likely to occur which, as in rolling cylinders, will alter to some extent their angular velocity ratio.

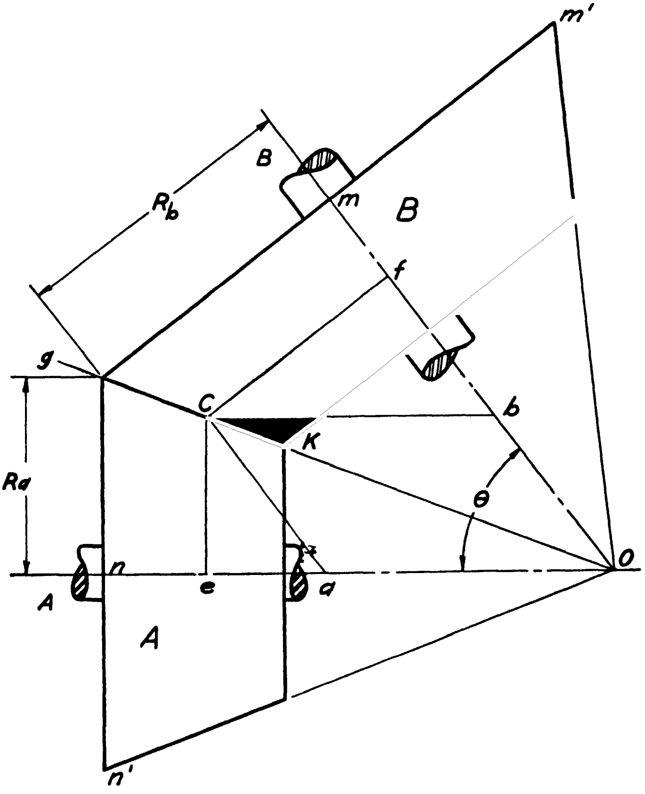


Fig. 109 Bevel Friction Wheels

An inspection of Fig. 109 shows that when two cones, *A* and *B*, are used to connect intersecting shafts, the apex of each cone is located at the point of intersection of the axes of those shafts, and the cones are in contact along a common element which lies in their common tangent plane. Any point of this line of contact, such as point *g*, is the point of contact of two circles which are transverse sections of the cones. However, these circular transverse sections do not lie in the same plane, as is the case with rolling cylinders. A common tangent to the two circles in contact at g will be perpendicular to the plane of projection used in Fig. 109. It is therefore perpendicular to the line of contact, Kg, and to mg and ng, radii of the circles in contact at g. Therefore the motion of this point of contact g, considered as a point of either circle, will be along this tangent and, with no slipping, the velocity of g as a point of cone A is equal to the velocity of g as a point of cone B. Let

V_L = the equal velocities of the points of contact at g

$R_a = ng$, the contact radius of cone A

$R_b = mg$, the contact radius of cone B

V_{aa} = the angular velocity of cone A

V_{ab} = the angular velocity of cone B

From formula (16),

$$V_L = R_a V_{aa}$$

and

$$V_L = R_b V_{ab}$$

Therefore

$$R_a V_{aa} = R_b V_{ab}$$

Dividing both members of the equation by $R_a V_{ab}$,

$$\frac{V_{aa}}{V_{ab}} = \frac{R_b}{R_a}$$

Since the ratio of the angular velocities is equal to the ratio of the r.p.m.'s,

$$\frac{N_a}{N_b} = \frac{R_b}{R_a} \quad \text{(Compare with formula [39])}$$

Hence with rolling cones, the angular velocities or r.p.m.'s are inversely proportional to the contact radii.

Graphical Construction of Bevel Frictions. Let it be required to construct two cones that may be used to connect a pair of intersecting shafts, knowing the angle of intersection, θ, and the r.p.m.'s of the shafts, N_a and N_b. Referring to Fig. 109, the axes of the shafts are laid out as shown, intersecting at point O. From the latter point, distances Oa and Ob are taken on the axes of A and B respectively, so that $\frac{O_a}{O_b} = \frac{N_a}{N_b}$. Lines aC and bC are next drawn parallel to the axes of the shafts so that a parallelogram, $OaCb$, is formed. The diagonal OC of this parallelogram is now drawn. This diagonal is the common contact element of the pair of rolling cones that are to be used under the conditions as given. Therefore the cone angles are COf and COe. To locate a pair of frusta of these cones, take any point g on the diagonal. Lay off any distance gK to represent the common contact element of the frusta. From both g and K drop perpendiculars to the axes of A and B, extending these perpendiculars beyond the axes (as in the case of gm to m' so that $mm' = gm$) to form the completed frusta, A of shaft A and B of shaft B.

That these cones will give the desired speed ratio can be proved in the following manner. From any point such as C on the line of contact, drop perpendiculars, Ce and Cf to the axes of A and B respectively. In the similar right triangles, aCe and bCf,

$\frac{eC}{fC} = \frac{aC}{bC}$ (Corresponding sides of similar triangles are proportional.)

But $\frac{aC}{bC} = \frac{Ob}{Oa}$ (Opposite sides of a parallelogram are equal.)

Since $\frac{Ob}{Oa} = \frac{N_b}{N_a}$ (By construction)

Therefore $\frac{eC}{fC} = \frac{N_b}{N_a}$ (Things equal to the same thing are equal to each other.)

The distances eC and fC are in reality corresponding radii of the cones A and B, respectively, for they are drawn to the same point on the common contact element. Therefore it is proved that the radii of the cones are inversely proportional to the r.p.m.'s, and hence will give the desired angular velocity ratio. However, the cones are given by their outer or base radii. It may be noted therefore that since triangle Ogm is similar to triangle OCf, and triangle Ogn is similar to triangle OCe,

$$\frac{ng}{mg} = \frac{eC}{fC} = \frac{N_b}{N_a}$$

But ng and mg are the radii of the bases of A and B respectively, therefore

$$\frac{N_b}{N_a} = \frac{R_a}{R_b}$$

or $\frac{N_a}{N_b} = \frac{R_b}{R_a}$ as given in formula (39)

Brush Wheel and Plate. A friction drive of the form shown in Fig. 110 is known as the *Brush Wheel and Plate.* The brush wheel, A, is located on shaft S_1. The plate, B, is located on shaft S_2. The two shafts are at right angles to each other so that the axis of A, or its shaft, is parallel to the surface of B. The wheel A is mounted in such a manner that it may be shifted axially along its shaft into different positions of contact with B. Either A or B may be used as the driver. Assuming that the wheel A is the driver and that its rotation is uniform, it will be noted that the angular velocity of B will constantly increase as A is shifted nearer the axis of B. Naturally the converse is true if A is shifted away from that axis. If it is carried to the opposite side of the axis, the direction of rotation of the plate will be reversed. Thus this mechanism affords a variable speed ratio for either direction of rotation of the follower.

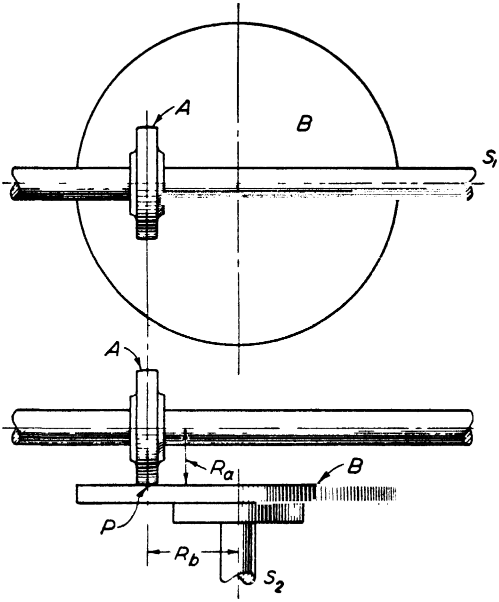


Fig. 110 Brush Wheel and Plate

As with all friction drives, the velocity ratio between driver and follower in this mechanism is based on the assumption of pure rolling. Such a condition with this mechanism would necessitate point contact, whereas in its practical application it has line contact. Point contact however can be approached either by using a thin wheel or by rounding off its surface so that contact will be in its mid-plane.

Referring to Fig. 110, let R_a be the constant radius of the brush wheel and R_b the radius of the plate for the position of the wheel as shown. Applying formula (**11**), to give us V_{La} and V_{Lb}, the linear velocities of P_a and P_b respectively,

$$V_{La}=2\pi R_aN_a$$
$$V_{Lb}=2\pi R_bN_b$$

Since in pure rolling the linear velocities of the points of contact are equal,

$$V_{La}=V_{Lb}$$

therefore $$2\pi R_aN_a=2\pi R_bN_b$$

Dividing both members of this equation by 2π

$$R_aN_a=R_bN_b$$

Dividing now by N_aR_b,

$$\frac{R_a}{R_b}=\frac{N_b}{N_a}$$

Note that the above formula is the same as formula (**39**)

Multiplying by N_a,

$$N_b=\frac{R_a}{R_b}\times N_a \qquad (42)$$

The proportion given by formula (39) can be used of course in solving for any one of its elements when the other three are known. Formula (42) is easily applied in solving for the r.p.m. of B when A is the driver with its constant r.p.m. N_a, its constant radius R_a, and with R_b, its position relative to B, given.

Dividing both members of formula (42) by $\frac{R_a}{R_b}$,

$$N_a=\frac{N_b}{\frac{R_a}{R_b}}$$

Therefore $$N_a=\frac{R_b}{R_a}\times N_b \qquad (43)$$

Formula (43) is easily applied in solving for the r.p.m. of the brush wheel A when the plate is the driver and hence its r.p.m. is given.

Example 1. Two shafts are connected by a brush wheel and plate. The brush wheel is 4 inches in diameter and has a speed of 250 r.p.m. How far from the center of the plate must the brush wheel be located if the plate makes 50 r.p.m.

Solution. Referring to Fig. 110, $N_a = 250$ r.p.m., $N_b = 50$ r.p.m., and $R_a = \frac{4}{2} = 2$ in. It is required to find R_b.

From formula (39)

$$\frac{R_b}{R_a} = \frac{N_a}{N_b}$$

Evaluating $$\frac{R_b}{2} = \frac{250}{50}$$

Therefore $$R_b = \frac{250}{50} \times 2 = 10 \text{ in.} \quad \textit{Ans.}$$

Example 2. A plate having a speed of 90 r.p.m. drives a brush wheel whose diameter is 4.5 inches. What is the rotative speed of the brush wheel when it is located at a distance of 7.5 inches from the center of the plate.

Solution. Here $N_b = 90$ r.p.m., $R_b = 7.5$ in., and $R_a = \frac{4.5}{2} = 2.25$ in

Applying formula (43)

$$N_a = \frac{R_b}{R_a} \times N_b$$

and evaluating therein,

$$N_a = \frac{7.5}{2.25} \times 90$$

$$= 3\tfrac{1}{3} \times 90 = 300 \text{ r.p.m.} \quad \textit{Ans.}$$

Example 3. Find the linear velocity of the point of contact of the preceding example in feet per minute, assuming the point of contact first as P_a, a point of the wheel, and then as P_b, a point of the plate.

Solution. To find the velocity of P_a, $N_a = 300$ r.p.m. and $R_a = \frac{2.25}{12}$ ft. Applying formula (11)

$$V_{La} = 2\pi R_a N_a$$

and evaluating therein,

$$V_{La} = 2 \times \pi \times \frac{2.25}{12} \times 300$$

$$= 353.43 \text{ f.p.m.} \quad \textit{Ans.}$$

To find the velocity of P_b, $N_b = 90$ r.p.m. and $R_b = \frac{7.5}{12}$ ft. Applying formula (11)

$$V_{Lb} = 2\pi R_b N_b$$

and evaluating therein

$$V_{Lb}=2\times\pi\times\frac{7.5}{12}\times 90$$

$$=353.43 \text{ f.p.m.} \quad \textit{Ans.}$$

QUESTIONS AND PROBLEMS

1. State the three types of action that may occur between the acting surfaces of the driver and follower in a direct-contact mechanism.

2. What is the so-called line of action in a direct-contact mechanism? Through what point must it pass?

3. What relationship exists between the normal components of the linear velocities of the points of contact?

4. State the angular velocity theorem for a direct-contact mechanism.

5. State the condition of constant angular velocity ratio between driver and follower.

6. At the intersection of what two lines is the instant center of the relative motion of the driver with respect to the follower always located in a direct-contact mechanism?

7. How is the rate of sliding between the acting surfaces of driver and follower determined?

8. What is the relationship between the tangential components of the velocities of the contact points when the rate of sliding equals zero?

9. State the condition of pure rolling.

10. What is the location of the instant center of the relative motion of driver and follower when pure rolling takes place?

11. What relationship exists between the linear velocities of the points of contact during pure rolling?

12. Under what conditions will the angular velocities of driver and follower be inversely as the contact radii?

13. State the three conditions for pure rolling between ellipses.

14. State the condition of positive driving.

15. Do rolling circles (or right circular cylinders) give positive driving?

16. Can a constant angular velocity ratio be obtained by the use of a pair of rolling circles?

17. In Fig. 98(*a*), what is the r.p.m. of *B* if the diameter of *A* is 8 inches, the diameter of *B* is 20 inches, and the rotative speed of *A* is 225 r.p.m.? *Ans.* 90 r.p.m.

18. In the preceding problem, it is required to find:

(*a*) V_{La}, the linear velocity of the point of contact P_a, in f.p.m.

(*b*) V_{Lb}, the linear velocity of the point of contact P_b, in f.p.m.

(*c*) V_{aa}, the angular velocity of *A*, in radians per second.

(*d*) V_{ab}, the angular velocity of *B*, in radians per second.

(*e*) the value of the angular velocity ratio, $\frac{V_{aa}}{V_{ab}}$

(*f*) the value of the r.p.m. ratio, $\frac{N_a}{N_b}$

Ans. (*a*) 471.24 f.p.m. (*b*) 471.24 f.p.m. (*c*) 23.56 radians per sec. (*d*) 9.42 radians per sec. (*e*) 2.5 (*f*) 2.5

19. Construct an ellipse whose major and minor axes are 3 inches and 2 inches in length respectively, and locate the foci thereon.

20. State the kind of a transverse section that members of any friction drive must have.

21. Two parallel shafts, A and B, rotating in opposite directions, are to be connected by a pair of spur friction wheels (rolling cylinders). It is required to find the diameters of the cylinders to be used if the r.p.m. of A is 200, the r.p.m. of B is 75, and the distance between centers is 11 inches. *Ans.* 6 in.; 16 in.

22. Two parallel shafts, A and B, rotating in the same direction, are to be connected by a pair of spur friction wheels. It is required to find the diameter of the cylinders to be used if the r.p.m. of A is 300, the r.p.m. of B is 120, and the distance between centers is 9 inches. *Ans.* 12 in.; 30 in.

23. It is required to construct a pair of bevel frictions (rolling conical frusta) to connect two intersecting shafts, A and B, the following data being given: $N_a = 325$ r.p m., $N_b = 150$ r.p.m., and θ, the angle at which the axes of the shafts intersect, $= 90$ degrees.

24. Two shafts are connected by a brush wheel and plate. The brush wheel is 5 inches in diameter and has a rotative speed of 200 r.p.m. How far from the center of the plate must the wheel be located if the plate makes 60 r.p.m. *Ans.* 8⅓ in.

25. Find the linear velocity of the point of contact of the mechanism of the preceding problem, assuming the point of contact first as P_a, a point of the wheel, and then as P_b, a point of the plate. *Ans.* 261.8 f.p.m.

26. Two shafts are connected by a brush wheel and plate. The brush wheel is 6 inches in diameter, makes 225 r.p.m., and is located at a distance of 10 inches from the axis of the plate. What is the rotative speed of the plate? *Ans.* 67.5 r.p.m.

GROUP OF ROLLING MILL PINIONS AND GEARS
Courtesy of The Falk Corporation, Milwaukee Wis

CHAPTER VIII

TOOTHED WHEELS OR GEARS

Introduction. It has been noted in the preceding chapter that, in all so-called friction drives, the transmission of motion from the driver to the follower is dependent on friction between the acting surfaces, since such mechanisms do not afford positive driving. Hence some slippage may result, during which the pure rolling action of driver and follower is destroyed and their angular velocity ratio ceases to be constant. There are many cases in machine design where a constant angular velocity ratio must be maintained by a direct-contact mechanism throughout its cycle, with no possibility whatsoever of any deviation from that velocity ratio. It is evident that in such cases positive drives, instead of friction drives, must be used. Hence outlines for the acting surfaces must be procured which will keep the common normal from passing through either the fixed center of driver or follower, and which at the same time will cause the common normal to cut the line of fixed centers at the same point throughout the cycle. In other words, the acting surfaces must be given outlines which will simultaneously satisfy both the condition of positive driving and the condition of constant angular velocity ratio.

Such positive-drive direct-contact mechanisms which will give a constant velocity ratio are in reality obtained through the alteration of the surfaces of friction drives, so that interlocking members, called teeth, are provided, and the resulting links are called *Toothed Wheels* or *Gears*. The former surfaces of the friction members, although completely destroyed, are retained in a sense as the basis of the design and measurement of the resulting gears, and are known as the *Pitch Surfaces* of the gears. The transverse sections of these surfaces are called the *Pitch Circles* of the gears, and in the design of gears these pitch circles are tangent to each other as they formerly were in friction drives. Toothed wheels, resulting from the alteration of cylindrical or spur frictions, and therefore used for connecting parallel shafts, are known as *Spur Gears*, see Fig. 111 (*a*) and (*b*); while those resulting from the alteration of bevel frictions, and therefore used to connect intersecting shafts, are known as *Bevel Gears,* see Fig. 144. The alteration is so effected that the gears retain the constant angular velocity ratio of their corresponding friction members. Their pitch surfaces will theoretically retain their pure rolling action, but the actual contact members, which are now the interlocking teeth, will have an action with each other which is a mixture of rolling and sliding.

It is evident from Fig. 111 (*b*) that two mating gears must have integral numbers of teeth and that, furthermore, the teeth of one gear must be so proportioned that they will fit into the spaces of the other.

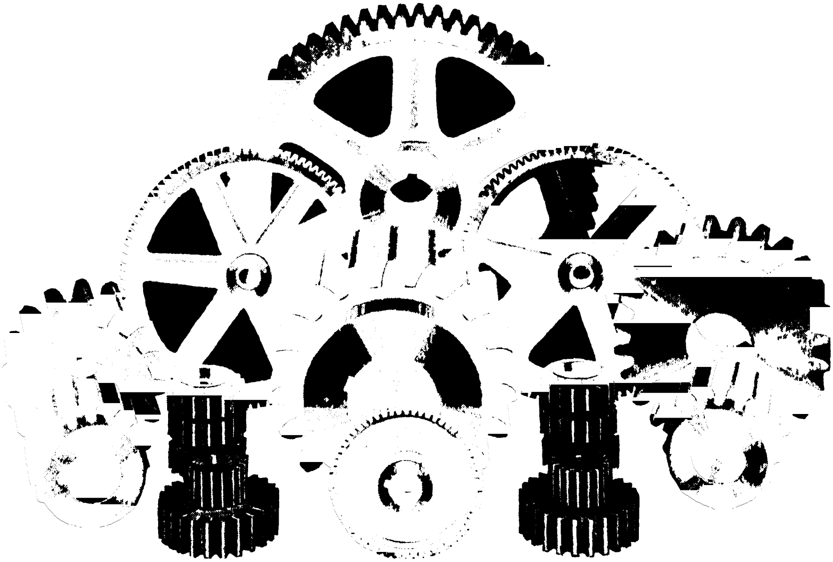


Fig 111(*a*) Spur Gears
Courtesy of Foote Bros Gear and Machine Corporation, Chicago Ill

The basic definitions and general principles presented in this chapter are based on spur gears, due to the fact that they are the simplest type of gears. However, such information is general and is therefore applicable to other gears as well.

Law of Gearing. It is evident from the preceding article that since the engaging teeth are to secure a constant velocity ratio for the gears, the outlines or profiles (see Figs. 114 (*a*) and 126) of these teeth must be so formed that the common normal to the profiles at the point of contact always intersects the line of centers at the same point.

As stated previously, the gears retain the angular velocity ratio of the friction members from which they were formed. It has been shown that this velocity ratio is the inverse ratio of the radii of the friction members, which radii have become the pitch circle radii (or pitch radii) of the gears. It follows then that the segments of the line of centers that are cut by the common normal must be these pitch radii. (Review the angular velocity theorem of Chapter VII.) Therefore the profiles of engaging gear teeth must be so formed that the common normal to them at their point of contact must always cut the line of centers of the gears at the point of tangency of their pitch circles, the so-called *Pitch Point.* This statement is known as the *Law of Gearing.* Tooth profiles which conform to this law are said to be *Conjugate* to each other.

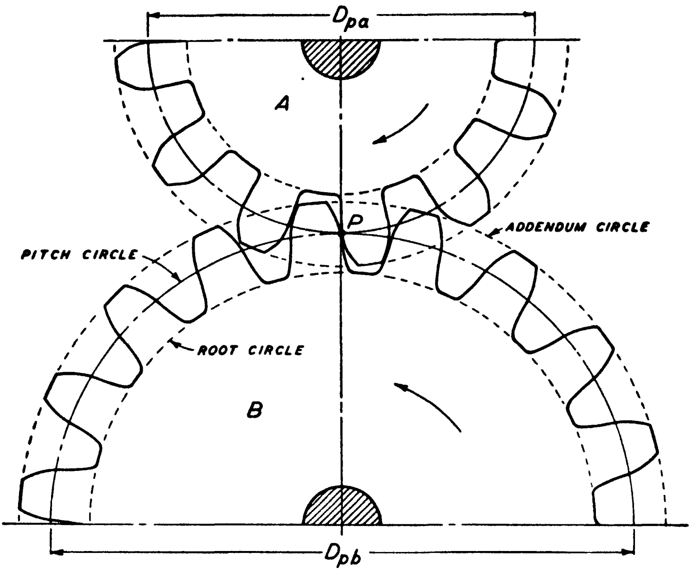


Fig. 111(*b*). Spur Gears

Curves Used for Gear-Tooth Profiles. There are a number of curves which can be so used for the outlines or profiles of gear teeth that they will obey the law of gearing. However, from a commercial standpoint only two are employed. These are the *Cycloidal* and *Involute* curves which give rise to the so-called *Cycloidal* and *Involute Systems.* These systems are used with gears of all types.

Cycloidal Curves. This is a family of curves that has three members, the names of which are: the *Cycloid*, from which the group obtains its name; the *Epicycloid*; and the *Hypocycloid*. Each of these curves is generated by a rolling circle, being the path of a point of the circle's circumference. If the circle rolls on a straight line, the path is a cycloid. If the circle rolls on the outside or convex side of another larger circle, the path is an epicycloid, while if it rolls on the inside or concave side, the path is a hypocycloid.

Construction of the Cycloid. In Fig. 112 the generating circle, G, is shown in contact with the line LL_1 at point P. As the circle rolls to the right of its initial position, point P takes successively the new positions, a, b, c, etc. A smooth curve PM connecting these points is the path of point P and is the cycloid generated by circle G. If the generating circle is allowed to roll upon the line through one complete revolution, point P will again be the point of contact of G and LL_1. Hence the cycloid, when completed, returns to LL_1 at a point which is at a distance equal to the circumference of G from the initial position of P.

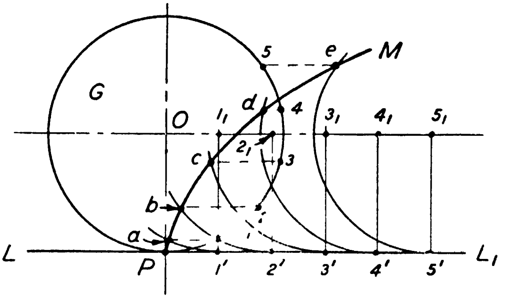


Fig. 112 Cycloid

In order to locate the points a, b, c, etc., of the cycloid PM, lay off, on the circumference of G, the equal arcs P-1, 1-2, 2-3, etc. A small arc length should be used so that its length is closely approximated by its chord. This same distance is then used in locating (to the right of P on line LL_1) the equally spaced points, $1'$, $2'$, $3'$, etc. Through these points, vertical lines are drawn intersecting the horizontal center line of G at 1_1, 2_1, 3_1, etc. It is evident that as G rolls upon LL_1, its center O moves along the horizontal center line. When O reaches such a point as 1_1, the generating circle will be tangent to

LL_1 at $1'$, and point 1 of the circle will be coincident with point $1'$. Likewise when O reaches the position 2_1, the circle will be tangent to LL_1 at $2'$ and point 2 of the circle will coincide with $2'$. Draw the circle in each of these several successive tangent positions. Then through point 1 on G draw a line parallel to LL_1 intersecting circle $1'a$ with center at 1_1 at point a. In a similar manner, lines drawn successively through the other points 2, 3, 4, and 5 on G, parallel to LL_1 intersect circles $2'b$, $3'c$, $4'd$, and $5'e$ at points b, c, d, and e, respectively.

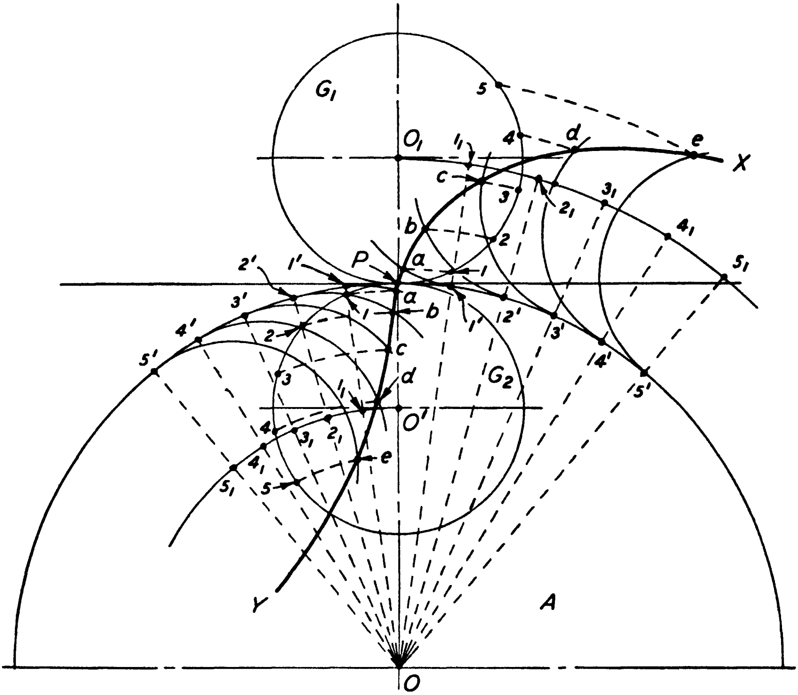


Fig. 113 Epicycloid and Hypocycloid

Construction of the Epicycloid. As the generating circle G_1 of Fig. 113 rolls on the outside of circle A to the right of its initial position, it generates the epicycloid PX. The latter is the path of P which is the point of tangency of circles G_1 and A. It is therefore defined by the successive positions P, a, b, c, etc., which the point P takes as G_1 rolls on A.

The construction of the epicycloid is practically the same as that of the cycloid. In order to locate the points a, b, c, etc., such points as

1, *2*, *3*, and *1′*, *2′*, *3′* are laid off at equal distances along their respective circles. The path $O_1 5_1$ of center O_1 of G_1 is described with O as a center and OO_1 as a radius. Radial lines are then drawn from the center O of circle A, through points *1′*, *2′*, *3′*, etc., intersecting $O_1 5_1$ at 1_1, 2_1, 3_1, etc. With the latter points as centers, the successive positions of the generating circle, G_1, are drawn. Now with O as a center, concentric arcs are described through points *1*, *2*, *3*, etc., which are located on the initial position of G_1. These arcs intersect, in the order as given, circle *1′a* at point *a*, circle *2′b* at *b*, circle *3′c* at *c*, circle *4′d* at *d*, and circle *5′e* at *e*. These points, *P*, *a*, *b*, *c*, *d*, and *e*, are connected by

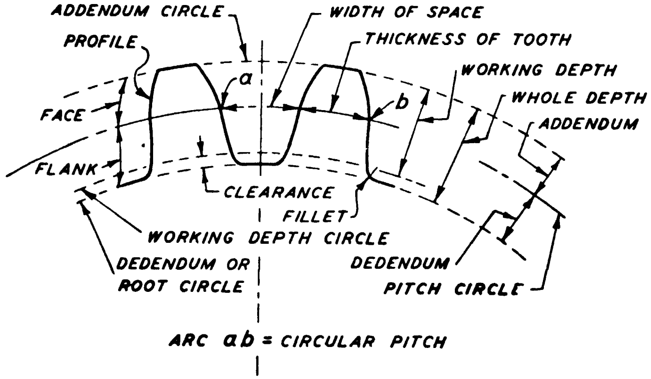


Fig 114(a)

a smooth curve, *PX*, which is that epicycloid of *A* generated by the rolling circle G_1.

Construction of the Hypocycloid. In Fig. 113 a circle, G_2, tangent to circle A at point P, rolls to the left of its initial position generating the hypocycloid *PY*. The method by which it is constructed is identical to that used in obtaining *PX*, the same notation being used throughout. It should be noted, however, that the path of the center O' of circle G_2 is the circle $O'5_1$, which is drawn with O as a center and OO' as a radius. Hence the successive positions of the generating circle are tangent internally to circle A. In Fig. 113 circle G_2 could have been rolled to the right just as well as to the left. However when an epicycloid and a hypocycloid of the pitch circle of a gear are used as the gear-tooth profile, their generating circles are rolled in opposite directions.

Definitions of Terms Used with Gears. See Fig. 114 (*a*) and (*b*). The *Pitch Circle* is the most important circle of a gear. The size of a gear is dictated by the diameter of the pitch circle. Thus a 12-inch gear is one whose pitch circle has a diameter of 12 inches. This diameter is referred to as the *Pitch Diameter.* The pitch circle is the circle on which, and from which, practically all the dimensions of the gear teeth are taken. When one gear is in mesh with, or driving, another, their pitch circles are tangent to each other.

The *Pitch Point* of a pair of gears is the point of tangency of their pitch circles.

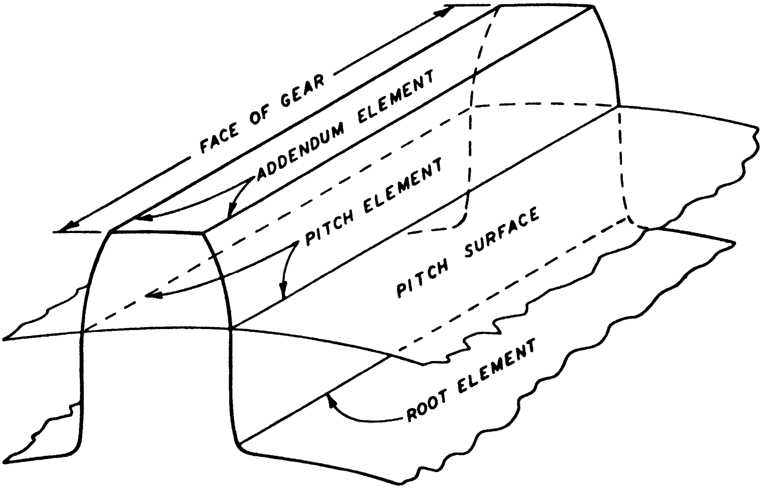


Fig 114(*b*)

The *Addendum Circle* is the circle that bounds or limits the height of the teeth. It is the outermost circle of the gear and its diameter is often called the outside diameter.

The *Root* or *Dedendum Circle* is the circle from which the teeth protrude or extend. It bounds the bottom of the teeth and hence the spaces between the teeth.

The *Base Circle* is one that occurs only in the involute system of gears. It is the circle from which the involute curve of the tooth profile is generated. (See Fig. 126.)

The *Circular* or *Circumferential Pitch* is the distance measured in inches from a point on one tooth to the corresponding point on the next or adjacent tooth, the distance being measured along, and hence being an arc of, the pitch circle. It is evident that any two gears which will mesh or work together as driver and follower must have the same circular pitch in order that the teeth of one gear may fit or mesh into the spaces of the other.

The *Diametral* (di-am′e-tral) *Pitch* is the ratio of the number of teeth to the number of inches in the pitch diameter. In other words, it is the number of teeth per inch of diameter of the gear. Thus a 2-pitch gear is one whose diametral pitch is 2, and therefore it has 2 teeth for every inch of its pitch diameter. Gears which work together must have the same diametral pitch.

The *Thickness of Tooth* is its thickness measured along the pitch circle.

The *Width of Space* is its width measured along the pitch circle. It is evident that the thickness of tooth plus the width of space is equal to the circular pitch.

Backlash is the difference between the width of space and the thickness of tooth. Theoretically, for well-formed or cut teeth, the width of space is equal to the thickness of tooth, and hence the backlash is equal to zero. But from a practical standpoint, the width of space is made slightly greater than the thickness of tooth to provide for irregularities in the form or spacing of the teeth, so that there will be no binding as the tooth of one gear enters the space of the other.

The *Addendum* is the radial distance from the pitch circle to the addendum circle.

The *Dedendum* or *Root* is the radial distance from the pitch circle to the dedendum or root circle.

The *Pitch Element* of a tooth is the line along which the pitch surface is intersected by the tooth. It is evident that the pitch element as well as any other element of a spur-gear tooth is a straight line which is parallel to the axis of the gear. See Fig. 114 (*b*) for the addendum and root elements.

The *Whole Depth* of a tooth is the radial distance from the addendum circle to the dedendum circle. It is evidently equal to the sum of the addendum and dedendum distances.

The *Clearance* is a radial distance equal to the difference obtained by subtracting the addendum from the dedendum. It is the distance between the top of the mating tooth and the bottom of the space of the other gear into which the tooth projects.

The *Working Depth* is equal to two times the addendum.

The *Face of the Tooth* is that part of the profile between the pitch and addendum circles.

The *Face of the Gear* is its width measured along a tooth element of a straight-tooth gear.

The *Flank of the Tooth* is that part of the profile between the pitch and dedendum circles.

Pinion is a term which is commonly used to designate the smaller gear of a mating pair. In such a case the pair is referred to as a *Gear and Pinion.*

Relation between Circular and Diametral Pitches. Let

P_c = the circular pitch
P_d = the diametral pitch
D_p = the pitch diameter
t = the number of teeth

From the definition of circular pitch,

$$tP_c = \pi D_p \tag{44}$$

from which

$$t = \frac{\pi D_p}{P_c} \tag{45}$$

Since any gear must have an integral number of teeth, it follows from formula (45) that the circular pitch is contained in the pitch circumference, πD_p, a whole number of times.

From the definition of diametral pitch,

$$P_d = \frac{t}{D_p} \tag{46}$$

Multiplying both members of formula (46) by D_p

$$t = P_d D_p \tag{47}$$

Since the values of t as given in formulas (45) and (47) are equal,

$$P_d D_p = \frac{\pi D_p}{P_c}$$

Dividing both members of the above equation by D_p,

$$P_d = \frac{\pi}{P_c} \tag{48}$$

from which

$$P_c = \frac{\pi}{P_d} \tag{49}$$

Formulas (48) and (49) show that π divided by one pitch yields as its quotient the other pitch.

From formula (44), by dividing both members of the equation by π,

$$D_p = \frac{tP_c}{\pi} \tag{50}$$

This formula states that the diameter of a pitch circle of a gear is equal to the product of the number of teeth and the circular pitch divided by 3.1416.

Example 1. Find the circular pitch of a 4-pitch gear.

Solution. Here $P_d = 4$. From formula (49)

$$P_c = \frac{\pi}{P_d} = \frac{3.1416}{4} = 0.7854 \text{ in.} \quad \textit{Ans.}$$

Example 2. Obtain the number of teeth on a gear whose pitch diameter and diametral pitch are 8 inches and 3½ respectively.

Solution. Here $D_p = 8$ in., and $P_d = 3\frac{1}{2}$. From formula (47)

$$t = P_d D_p = 3\tfrac{1}{2} \times 8 = 28 \text{ teeth.} \quad \textit{Ans.}$$

Example 3. A 40-tooth spur gear has a circular pitch of 1¼ inches. Find the pitch diameter of the gear.

Solution. Here $t = 40$ teeth and $P_c = 1\frac{1}{4}$ in. Applying formula (50)

$$D_p = \frac{tP_c}{\pi}$$

$$= \frac{40 \times 1\frac{1}{4}}{\pi} = \frac{50}{\pi} = 15.915 \text{ in.} \quad \textit{Ans.}$$

Cycloidal Gear-Tooth Theory. In Fig. 115, gears A and B, (given by their pitch circles, A and B) with centers O_a and O_b, respectively, have pitch circles which are tangent to each other at point P. Line TT_1 is the common tangent to the pitch circles and hence is perpendicular to the line of centers, O_aO_b, at P. Circle G_1 with its center at O_1 is tangent to both A and B at P, as is also circle G_2, whose center is at O_2. It should be noted that this point of tangency, P, is the instant center of the relative motion of any pair of circles for which it is the point of tangency.

It is evident from the condition of pure rolling that G_1 can roll with both A and B. Since it rolls on the inside of A and on the outside of B, G_1 will generate a hypocycloid of A and an epicycloid of B. G_2 can also roll with A and B, generating, however, an epicycloid of A and a hypocycloid of B.

Now let G_1 roll with A and B as all three circles rotate about their respective centers, until point P of G_1, moving along the arc Pg_1 occupies some position such as g_1. Then point P of A, moving along its arc Pa_1, will be at a_1; and arc Pa_1 must be equal to arc Pg_1 for, with pure rolling, no slipping takes place.

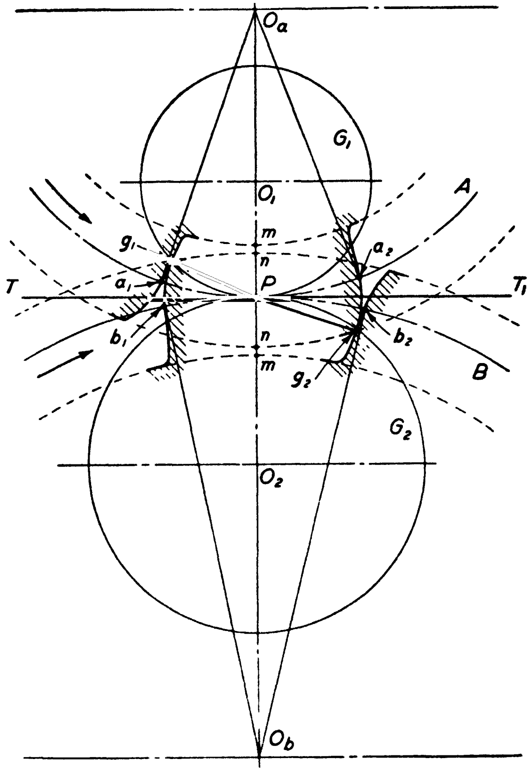


Fig 115

Before the rolling action started, a_1 was coincident with g_1 at P. Therefore the line a_1g_1 of our figure is the path that g_1 takes on A as G_1 rolls with A; or, in other words, line a_1g_1 is the hypocycloid of A generated by G_1.* In a similar manner, as P of A takes the position g_1, P of B, moving along its arc Pb_1, will arrive at b_1 and arc Pb_1 will be equal to arc Pg_1.

---

* **Note.** The hypocycloid in this case is not fixed while being generated, because it is on circle A, and A is rotating about its center. In Fig. 113, A is held stationary so that the hypocycloid remains fixed in position while being generated. This facilitates its construction.

Before the rolling action started, b_1 was coincident with g_1 at P. Therefore line b_1g_1 is the path that g_1 takes on B as G_1 rolls with B; or, in other words, line b_1g_1 is the epicycloid of B, generated by g_1 of circle G_1.

At any instant the direction of a line is of course the direction of motion of the point by which it is being generated, and the tangent to the line takes this direction. Therefore both lines a_1g_1 and b_1g_1, generated simultaneously by the same point, g_1, have the direction of the instantaneous motion of g_1 and are tangent to each other at g_1.

The instant center of the relative motion of G_1 with respect to both A and B is at point P, so that the instant radius of g_1 is Pg_1. It follows that the direction of motion of g_1, and hence of lines a_1g_1 and b_1g_1, is along a perpendicular to Pg_1 at g_1. Hence the common tangent to a_1g_1 and b_1g_1 is perpendicular to Pg_1 at g_1, which causes their common normal to coincide with Pg_1 and cut the line of centers O_aO_b at P. If a_1g_1 is now used as the flanks of the teeth of A, and b_1g_1 is used as the faces of the teeth of B, any point of contact between such a face and flank would be a position of the common generating point g_1 and hence the common normal would cut the line of centers at the pitch point P throughout the action of this epicycloidal face and hypocycloidal flank. It is therefore evident that cycloidal curves will conform to the law of gearing and so may be used for gear-tooth profiles when the faces and flanks that come into contact are generated by the same rolling circle.

With the direction of motion of the gears as given in Fig. 115, the teeth come into contact at g_1 and, as motion proceeds, the successive contact points draw more closely to the pitch circles until finally points a_1 and b_1 are in contact at P, which is the only phase at which the teeth will have pure rolling.

In order to carry the action to the right of point P, faces for the teeth of A and flanks for the teeth of B must be provided. Another generating circle, G_2, can evidently be employed for this purpose. This circle, rolling on the outside of A and the inside of B, generates the epicycloidal face a_2g_2 of A, and the hypocycloidal flank b_2g_2 of B; and arc $Pg_2 =$ arc $Pb_2 =$ arc Pa_2. This face a_2g_2 of A and flank b_2g_2 of B complete the contacting portion of the two profiles and permit the action of the teeth to continue beyond point P to point g_2, at which the teeth quit contact.

Every point of contact between a_2g_2 and b_2g_2 will be a position of the common generating point g_2, and hence during this action the common normal at each successive point of contact will pass through the pitch point P.

Since g_1 and g_2 are taken as the initial and final points of contact, it is evident that the teeth of A and B must be high enough to provide such points. Hence the addendum circle of A passes through g_2, and the addendum circle of B passes through g_1. It is thus seen that the intersections of the addendum circles and the generating circles are the initial and final points of contact in the action of a pair of cycloidal gear teeth.

The flanks of the teeth of A and B must extend inward beyond g_1 and g_2 respectively, so that the top of the tooth of one gear will not make contact with the bottom of the space of the other. This so-called clearance is made possible by locating the root circle of one gear so that it fails to contact the addendum circle of the other. In Fig. 115 this clearance provided is the distance mn.

For the sake of clarity, the completed profiles are shown in the figure. The same profiles are used on the opposite side (not shown in the figure) of the teeth so that driving is afforded in either direction.

Angle and Arc of Action. See Fig. 115. The central angle of the pitch circle through which a gear sweeps while one of its teeth is in continuous contact with a tooth of the mating gear, is called the *Angle of Action* of the gear. It will be noted in the figure that when the teeth are at their initial point of contact, g_1, the tooth profile of A cuts its pitch circle at a_1 while the tooth profile of B cuts its pitch circle at b_1, and that when contact between the teeth ceases at point g_2, the tooth profile of A is at a_2 while the tooth profile of B is at b_2. The *Angle of Approach* of a gear is the central angle of its pitch circle through which the gear turns while one of its teeth is in contact from the initial point of contact to the pitch point. The *Angle of Recess* is the central angle through which the gear turns while one of its teeth is in contact from the pitch point to the point at which contact ceases. In the cycloidal gears of Fig. 115,

$a_1O_aa_2 =$ the angle of action of gear A
$b_1O_bb_2 =$ the angle of action of gear B
$a_1O_aP =$ the angle of approach of gear A
$b_1O_bP =$ the angle of approach of gear B

PO_aa_2 = the angle of recess of gear A
PO_bb_2 = the angle of recess of gear B

It is evident that the angle of action of a gear is equal to the sum of its angles of approach and recess.

The *Arc of Action* of a gear is that arc of its pitch circle which is subtended by the angle of action. The arc of action is made up of two arcs which are the *Arc of Approach* and the *Arc of Recess.* These are the arcs which are subtended by the angles of approach and recess. In the figure,

a_1Pa_2 = the arc of action of gear A
b_1Pb_2 = the arc of action of gear B
a_1P = the arc of approach of gear A
b_1P = the arc of approach of gear B
Pa_2 = the arc of recess of gear A
Pb_2 = the arc of recess of gear B

By construction, a_1P is equal to b_1P and Pa_2 is equal to Pb_2. Therefore

$$a_1Pa_2 = b_1Pb_2$$

or the arcs of action of any two mating gears are equal. It follows from this statement that their angles of action cannot be equal unless the gears are of the same size.

Obviously, when one pair of teeth is quitting contact at g_2, another pair must have already made contact, or at least be ready to make contact, at g_1, so that the driving will continue. Therefore the arc of action can never be less than the circular pitch. If the arc of action is equal to the circular pitch, one pair of teeth will cease contact just as the next pair makes contact. If the arc of action is twice the circular pitch, two pairs of teeth will always be in contact.

It has been found to be good practice to have the arc of action at least 1.4 times the circular pitch if the design will permit.

Path of Point of Contact. As stated previously, the successive points of contact throughout the angles of approach in Fig. 115 are positions of g_1, for G_1 simultaneously generated the hypocycloidal flank of A and the epicycloidal face of B. Since g_1 is a point on the generating circle G_1, it must always remain thereon; hence the *Path of the Point of Contact in Approach* is the arc g_1P. Likewise the successive points of contact throughout the angles of recess are positions of g_2, for G_2 simultaneously generated the epicycloidal face of A and the hypocycloidal flank of B; and g_2 is a point of the generating circle G_2 and must always remain on this circle. Hence the *Path of the Point of Contact in Recess* is the circular arc Pg_2. Throughout the entire action of a pair of teeth of A and B, the total *Path of the Contact Point* is then the reverse arc, g_1Pg_2.

Since by construction $Pg_1 = Pa_1 = Pb_1$ and $Pg_2 = Pa_2 = Pb_2$, the length of the path of contact is equal to the arc of action.

Angle of Obliquity or Pressure Angle. See Fig. 115. The angle between the common normal to the tooth profiles at any point of contact and the common tangent to the pitch circles is known as the *Angle of Obliquity* (ob-liq′ui-ty) or the *Pressure Angle.* Since, during approach, the point of contact moves along the arc g_1P, the pressure angle gradually decreases from a maximum at g_1 when the angle is g_1PT, to a minimum at P when the angle is equal to zero. In the latter case the common normal coincides with the common tangent TT_1. During recess, the point of contact moves along arc Pg_2 so that the pressure angle gradually increases from a minimum of zero at P to a maximum of g_2PT_1 at point g_2.

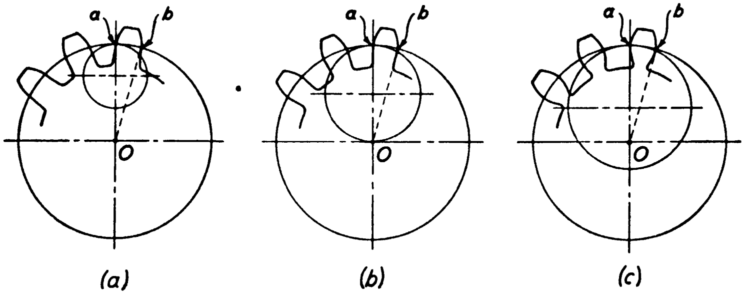
In the cycloidal system of gear teeth, the pressure angle is thus seen to be highly variable. This is a disadvantage, for it tends toward uneven wear of the tooth profiles.

Size of Generating Circle. For a pitch circle of a given diameter, the shape of any cycloidal curve depends upon the size of its generating circle. This fact is particularly important in the case of the hypocycloid, for the shape of the latter is a determining factor in the strength of the tooth and sometimes in the practical possibility of the tooth's construction.

The effect of the hypocycloid, and therefore its generating circle, upon the form of a tooth is brought out in Fig. 116. The gears of this figure are the same in every respect except that the hypocycloidal flanks of the teeth of each gear are generated by a rolling circle of a different size. In Fig. 116 (*a*) a rolling or generating circle is used, whose diameter is much less than the radius of the pitch circle, and the hypocycloidal flanks are seen to be directed away from, or outside of, the radii Oa and Ob. This produces a strong tooth, for it is relatively thick at its juncture with the root circle.

In Fig. 116 (*b*) the generating circle has a diameter equal to the radius of the pitch circle, and the hypocycloid formed coincides with the diameter of the pitch circle, so that the flanks of the tooth follow along the radii *Oa* and *Ob* and the tooth is said to have radial flanks. Evidently the teeth of this gear are not quite so strong as those of Fig. 116 (*a*).

The generating circle of the gear in Fig. 116 (*c*) is still larger than that of the preceding gear, so that its diameter is greater than the radius of the pitch circle. This causes the flanks of a tooth of this gear to lie inside the radial lines *Oa* and *Ob*, so that a weaker form of tooth is produced than in the two preceding gears. It is therefore seen that the larger the generating circle, under the conditions of Fig. 116, the weaker the teeth will be. This suggests that there must be some relationship between the diameters of the generating and pitch circles that should not be exceeded when any considerable amount of power is to be transmitted. The accepted relationship is a diameter of generating circle that is not more than five-eighths of the pitch diameter.



(a) (b) (c)

Fig 116

Interchangeable Set of Cycloidal Spur Gears. From the theory of cycloidal gears, previously given, the profiles of a single pair of meshing gears can be generated by two different rolling circles; for the only requirement is that the faces of the teeth of one gear and the flanks of the teeth of the other must be generated by the same rolling circle.

Let us suppose a pair of gears to be named *A* and *B* and that the faces of *A* and the flanks of *B* are generated by circle *1;* and the flanks of *A* and the faces of *B* are generated by circle *2*. Now let there be a third gear, *C*, having the same pitch as *A* and *B*. It is required that the profiles of the teeth of *C* be so generated that *C* may be paired with either *A* or *B*. If *C* is to work properly with *A*, its faces must be generated by circle *2* and its flanks by circle *1;* and if C is to work properly with B, its faces must be generated by circle *1* and its flanks by circle *2*. It is therefore obvious that circles *1* and *2* must be the same circle.

When three or more gears are so designed that any two of them may work properly together, they constitute what is called an *Interchangeable Set.* Hence it appears that an interchangeable set of cycloidal spur gears must not only have the same pitch, but must also have teeth whose faces and flanks are generated by the same (size of) rolling circle. The teeth of all gears of the set must of course be similarly proportioned.

The smallest gear of an interchangeable set is called the *Base* of the set. It is generally given either twelve or fifteen teeth. In order that the teeth of none of the set will be undercut or narrowed in at the base as those of Fig. 116 (*c*), the flanks of the teeth on the base of the set are made radial. Hence the diameter of the generating circle of the entire set must be equal to the radius of the pitch circle of the base of the set. If

P_d = diametral pitch of the set

P_c = circular pitch of the set

t = the number of teeth on the base of the set

R_p = the pitch radius of the base of the set

D_G = the diameter of the generating circle

then
$$2\pi R_p = tP_c$$

or
$$R_p = \frac{tP_c}{2\pi}$$

But
$$D_G = R_p$$

therefore
$$D_G = \frac{tP_c}{2\pi} \tag{51}$$

From formula (49)
$$P_c = \frac{\pi}{P_d}$$

Substituting this value of P_c in formula (51)

$$D_G = \frac{t \times \frac{\pi}{P_d}}{2\pi} = \frac{t \times \pi}{P_d} \times \frac{1}{2\pi}$$

$$D_G = \frac{t}{2P_d} \tag{52}$$

Example. Find the diameter of the generating circle that is used for an interchangeable set of cycloidal spur gears whose circular pitch is $1\frac{1}{4}$ inches. The base of the set is a pinion of 15 teeth.

Solution. Here $t=15$ teeth and $P_c=1\frac{1}{4}$ in. Applying formula (**51**)

$$D_G=\frac{tP_c}{2\pi}$$

and evaluating therein

$$D_G=\frac{15\times 1\frac{1}{4}}{2\pi}=2.98 \text{ in.} \quad \textit{Ans.}$$

Cast Gear Teeth. Gear teeth may be either cast or cut. Cast-tooth gears are made of cast iron, cast steel, cast steel alloys, or bronze. Manufacturers generally furnish gears of steel or steel alloys, either with or without heat treatment. If one gear of a pair of meshing gears is made of bronze, sparking is prevented. Cast-iron gears operating with cast-steel pinions tend to equalize resistance to wear. Cast teeth are not as reliable and free from noise as cut teeth, and hence are used for low speeds. Since the circular pitch is more convenient to the pattern maker, it is practically always used for cast gear teeth. It is generally made to vary by ⅛-inch increments.

It is evident from the statements previously made that teeth of a pair of meshing gears, whether cast or cut, must not only have the same circular pitch, but must also have the same tooth proportions. The tooth proportions or dimensions that are often used for cast gear teeth are as follows: (See Fig. 114 [*a*].)

Addendum $=0.32P_c$
Dedendum $=0.39P_c$
Addendum circle diameter $=D_p+(2\times$ the addendum$)$
Root circle diameter $=D_p-(2\times$ the dedendum$)$
Working depth $=2\times$ addendum $=0.64P_c$
Clearance $=$ dedendum $-$ addendum $=0.07P_c$
Whole depth $=$ addendum $+$ dedendum $=0.71P_c$
Thickness of tooth $=0.48P_c$
Width of space $=0.52P_c$
Backlash $=$ width of space $-$ thickness of tooth $=0.04P_c$
Pressure angle (for involute system) $=14\frac{1}{2}°$
Radius of fillet $=\frac{1}{7}$ the distance between teeth measured on the addendum circle.

Example 1. It is required to obtain the dimensions of the teeth of a cast spur gear whose circular pitch is ¾ inch.

Solution:

Addendum $= 0.32 \times 0.75 = 0.24$ in.

Dedendum $= 0.39 \times 0.75 = 0.2925$ in.

Working depth $= 0.64 \times 0.75 = 0.48$ in.

Clearance $= 0.07 \times 0.75 = 0.0525$ in.

Whole depth $= 0.71 \times 0.75 = 0.5325$ in.

Thickness of tooth $= 0.48 \times 0.75 = 0.36$ in.

Width of space $= 0.52 \times 0.75 = 0.39$ in.

Backlash $= 0.04 \times 0.75 = 0.03$ in.

Example 2. The gear of the preceding example has 40 teeth. Find (*a*) the pitch diameter, (*b*) the addendum circle or outside diameter, and (*c*) the root circle diameter of the gear.

Solution. (*a*) Here $P_c = \frac{3}{4} = 0.75$ in., and $t = 40$ teeth. Applying formula (44) (or formula [50])

$$\pi D_p = tP_c$$

and evaluating therein

$$\pi D_p = 40 \times 0.75$$

Dividing by π

$$D_p = \frac{40 \times 0.75}{\pi} = \frac{30}{\pi} = 9.549 \text{ in.} \quad \textit{Ans.}$$

(*b*) The diameter of the addendum circle is equal to the pitch diameter plus 2 times the addendum. Therefore

$$\text{the addendum circle diameter} = 9.549 + 2 \times 0.24$$
$$= 10.029 \text{ in.} \quad \textit{Ans.}$$

(*c*) The diameter of the root circle is equal to the pitch diameter minus 2 times the dedendum. Therefore

$$\text{the root circle diameter} = 9.549 - 2 \times 0.2925$$
$$= 8.964 \text{ in.} \quad \textit{Ans.}$$

Cut Gear Teeth. Cut gear teeth may be metallic or non-metallic. The metallic are made of cast iron, bronze, and carbon or alloy cast or forged steel, with or without special heat treatment or casehardening. The more carefully metallic gear teeth are cut, the smoother and hence the less noisy they will be in operation. To cut down the noise still further, so-called *Silent Gears* are made from materials such as Micarta, Bakelite, rawhide, etc. The base material of Micarta is a fabric woven to a definite weight and thickness. This fabric is carefully impregnated with a binder and is then subjected to heat and pressure. The result is a material which is claimed to produce silent, efficient and economical gears. (See Fig. 117.) Rawhide has long been used in manufacturing gears and pinions to work with metallic gears where quiet operation is desired. Gears made of such material as rawhide require metal flanges which reach to the top of the teeth and provide protection and support. (See Fig. 118.)

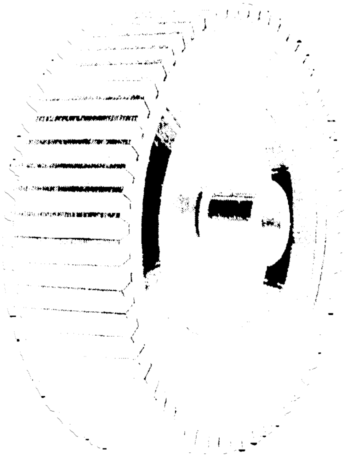


Fig 117 Non-Metallic Cut Spur Gear, Silent Gear with Steel Center
Courtesy of Foote Bros. Gear and Machine Corporation, Chicago, Ill

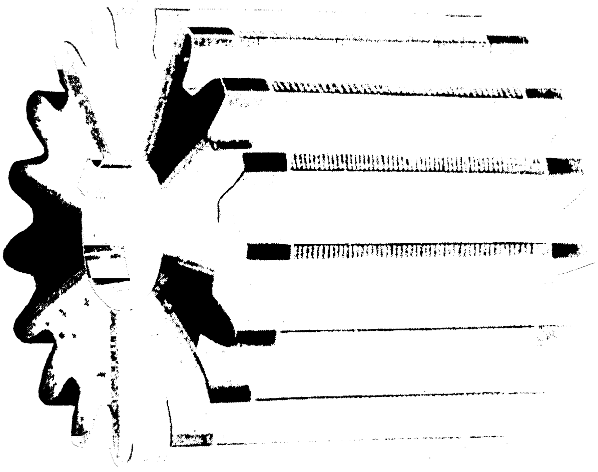


Fig 118 Non-Metallic Cut Spur Gear Note Brass Flanges and Laminations
Courtesy of Foote Bros. Gear and Machine Corporation, Chicago, Ill

There are several methods by which the cutting of gear teeth may be accomplished. Two of these methods are shown in Figs. 119 and 120. In Fig. 119 a spur gear is being cut by a form cutter, while in Fig. 120 a spur gear is being cut or generated by a cutter-gear.

Fig 119 Gear-Cutting Machine
Using Form Cutter

The use of a stocking cutter for roughing combined with a finishing cutter for finishing, is clearly shown on this machine

Courtesy of Brown & Sharpe Mfg Co, Providence, R I.

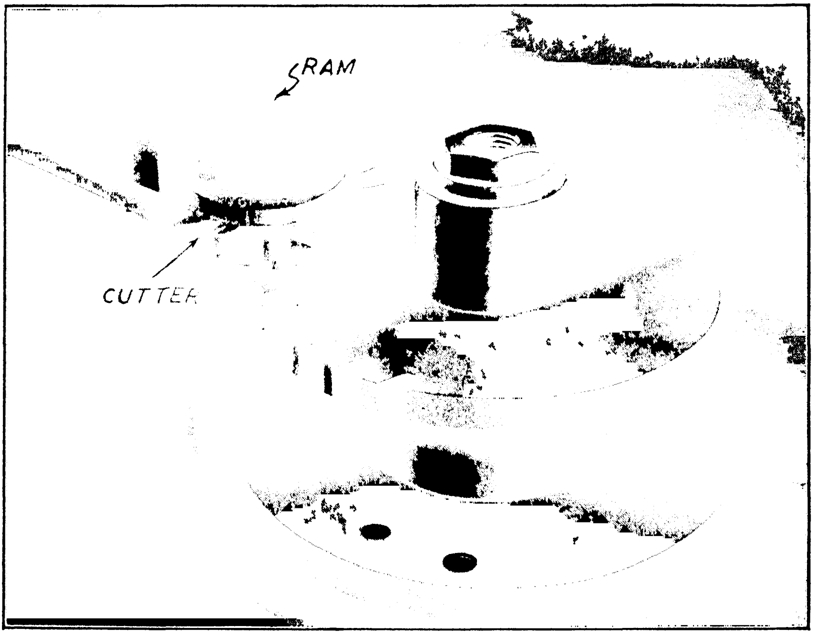


Fig 120 Cutting or Generating the Teeth of a Spur Gear

Courtesy of The Fellows Gear Shaper Co, Springfield, Vermont

The diametral pitch is the accepted pitch for cut gears. It has been standardized to vary

by $\frac{1}{4}$ increments from 1-pitch to 4-pitch
by $\frac{1}{2}$ increments from 4-pitch to 6-pitch
by 1 increments from 6-pitch to 16-pitch
by 2 increments from 16-pitch to 32-pitch

It should be noted that the larger the diametral pitch, the smaller the teeth.

Form cutters for each of the above pitches are obtainable for both the standard involute and cycloidal systems of gears. To cut a complete set of involute gears of a given pitch requires only one-third as many cutters as are required for the cycloidal system. This fact and the other inherent advantages of the involute system, which will be presented later, make the involute system dominate the field.

The tooth proportions for cut gear teeth of either the 14½-degree (full-depth) involute or cycloidal systems are as follows:

Addendum $= \dfrac{1 \text{ in.}}{P_d}$

Dedendum or root distance $= \dfrac{1.157 \text{ in.}}{P_d}$

Addendum circle diameter $= D_p + (2 \times \text{the addendum})$

Root circle diameter $= D_p - (2 \times \text{the dedendum})$

Working depth $= 2 \times \text{addendum} = \dfrac{2 \text{ in.}}{P_d}$

Clearance $= \text{dedendum} - \text{addendum} = \dfrac{0.157 \text{ in.}}{P_d}$

Whole depth $= \text{addendum} + \text{dedendum} = \dfrac{2.157 \text{ in.}}{P_d}$

Thickness of tooth $= 0.5\ P_c$

Width of space $= 0.5\ P_c$

Backlash $= 0$, theoretically

In practice, a small amount of backlash equal to about $\dfrac{0.04}{P_d}$ is actually used, which of course slightly changes the values given here for the thickness of tooth and the width of space.

Pressure angle (for involute system) = 14½ degrees

Radius of fillet = $\frac{1}{7}$ the distance between teeth, measured on the addendum circle.

Example. The following data is given for a cut spur gear: diametral pitch = 5½, number of teeth = 33.

It is required to find the pitch diameter, addendum circle diameter, root circle diameter, circular pitch, and the dimensions of the teeth of this gear.

Solution. Here $P_d = 5\frac{1}{2}$ and $t = 33$ teeth. Applying formula (47)

$$t = P_d D_p$$

and evaluating $$33 = 5\tfrac{1}{2} \times D_p$$

or $$D_p = \frac{33}{5\frac{1}{2}} = \frac{33}{\frac{11}{2}}$$

$$= 33 \times \tfrac{2}{11} = 6 \text{ in.}$$

Applying formula (49)

$$P_c = \frac{\pi}{P_d}$$

$$= \frac{\pi}{5.5} = 0.571 \text{ in.}$$

$$\text{Addendum} = \frac{1 \text{ in.}}{5.5} = 0.182 \text{ in.}$$

Addendum circle diameter $= D_p + (2 \times \text{the addendum})$
$= 6 + (2 \times 0.182) = 6.364$ in.

$$\text{Dedendum} = \frac{1.157 \text{ in.}}{5.5} = 0.210 \text{ in.}$$

Root circle diameter $= D_p - (2 \times \text{the dedendum})$
$= 6 - (2 \times 0.210) = 5.58$ in.

$$\text{Working depth} = \frac{2 \text{ in.}}{5.5} = 0.364 \text{ in.}$$

Clearance = dedendum − addendum
= 0.210 − 0.182 = 0.028 in.

Whole depth = addendum + dedendum
= 0.182 + 0.210 = 0.392 in.

Thickness of tooth = 0.5 × 0.571 = 0.286 in.

Width of space = 0.5 × 0.571 = 0.286 in.

Backlash = 0, theoretically.

Drawing Cycloidal Spur Gear Teeth. The method of drawing these teeth, presented in Fig. 121, is accomplished by means of generating the cycloidal tooth profiles. It is assumed that the number of teeth and circular pitch are known.

Draw the vertical and horizontal center lines of the gear *A*, calculate the pitch radius, and locate the pitch point *P* on the vertical center line at a distance from *O*, the center of the gear, equal to the pitch radius. Describe the pitch circle through point *P*. Assuming no backlash so that the thickness of tooth is equal to the width of space, divide the pitch circle into the equal divisions, *P-1*, *1-2*, *2-3*, etc., each of which is equal to one-half the circular pitch. It is evident that the number of these equal divisions will be equal to exactly twice the number of teeth. From the tooth proportions previously given in this chapter, compute the addendum and dedendum distances and describe the addendum and dedendum (root) circles.

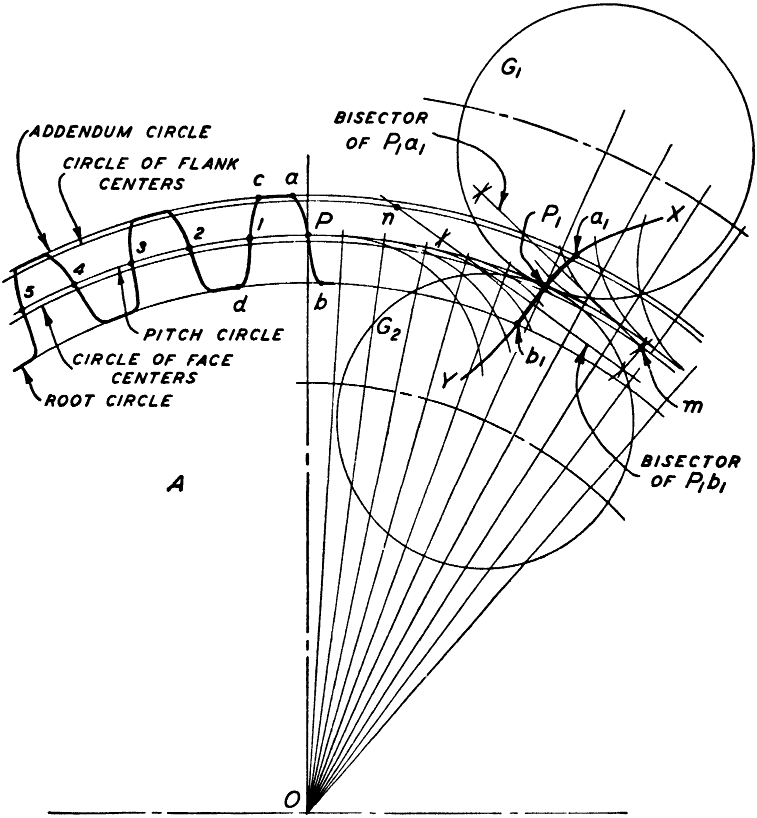


Fig. 121. Layout of Cycloidal Gear Teeth

At some convenient place on the pitch circle, such as P_1, lay out the generating circles G_1 and G_2, tangent to each other at P_1, and generate the epicycloid P_1X and the hypocycloid P_1Y. (See Fig. 113.) The addendum and dedendum circles intersect these curves at a_1 and b_1 respectively. Therefore $a_1P_1b_1$ is the tooth profile of the gear.

In order to draw this profile through the points P, 1, 2, etc., and thus complete the teeth, the approximate arc method will be used. This method uses circular arcs to approximate the epicycloidal face P_1a_1 and the hypocycloidal flank P_1b_1. It is necessary to obtain the radius and center of each of these approximate arcs. In order to do so, draw the perpendicular bisectors of the chords of curves P_1a_1 and P_1b_1, and upon these bisectors locate by trial the centers m and n of arcs which will practically coincide with these respective curves. Through points m and n describe circles concentric with the pitch circle. The circle through m is the one upon which the compass will be centered when drawing the faces of the teeth, while the circle through n is the one upon which the compass will be centered when drawing the flanks of the teeth. Now set the compass to the face radius, mP_1, and, centering it along the circle of face centers, describe arcs such as Pa through points P, 2, 4, etc., and arcs such as $1c$ through 1, 3, 5, etc. In a similar manner set the compass to the flank radius, nP_1, and, centering it along the circle of flank centers, describe arcs such as Pb through points P, 2, 4, etc., and arcs such as $1d$ through 1, 3, 5, etc. The profiles may now be completed by inserting the fillet at the bottoms of the spaces.

When another spur gear, say B, is to be shown in mesh with gear A of Fig. 121, the following principles must be observed.

1. The pitch circles of A and B will be tangent at the pitch point P.
2. The tooth profile of B must be generated in accordance with the law of gearing, and the face and flank radii determined in the same manner as for A.
3. The pitch and tooth proportions of B will of course be identical to those of A.

Grant's Cycloidal Odontograph. Another method of drawing cycloidal teeth, proposed by Mr. George B. Grant, employs a table (see Table I) in which are given, not only the radii of the circular arcs which will approximate the faces and flanks, but also the location of the circles of centers to be used in describing these arcs. Thus it is seen that this method eliminates the necessity of actually generating the tooth profile, so that the drawing of the teeth is greatly simplified.

In order to demonstrate the use of Table I, which is called Grant's Cycloidal Odontograph, the teeth of a 30-tooth 3-pitch spur gear will be drawn. Refer to Fig. 122. From the given data, determine the diameter of the pitch circle, the circular pitch, the addendum, and the dedendum.

Now describe the pitch, addendum, and dedendum circles. Locate points on the pitch circle at a distance from each other equal to one-half the circular pitch. In order to draw the tooth profiles through these points, the circles of centers and the face and flank radii are obtained from the odontograph table as follows: In the column headed "Number of Teeth," locate the number *30* (the number of the teeth on this gear) at the interval *30–36*. (This indicates that all gears having from 30 to 36 teeth, inclusive, employ the same data from the table.) To the immediate right of this is *2.40* inches, which, as stated at the top of the column, is to be divided by the

TABLE I—Grant's Cycloidal Odontograph

Number of Teeth in the Gear		For Diametral Pitch = 1				For Circular Pitch = 1 In			
		For Any Other Pitch, Divide by That Pitch				For Any Other Pitch, Multiply by That Pitch			
		Faces		Flanks		Faces		Flanks	
Exact	Intervals	Radius	Distance	Radius	Distance	Radius	Distance	Radius	Distance
10	10	1.99	.02	− 8.00	4.00	.62	.01	−2.55	1.27
11	11	2.00	.04	−11.05	6.50	.63	.01	−3.34	2.07
12	12	2.01	.06	∞		.64	.02	∞	
13½	13–14	2.04	.07	15.10	9.43	.65	.02	4.80	3.00
15½	15–16	2.10	.09	7.86	3.46	.67	.03	2.50	1.10
17½	17–18	2.14	.11	6.13	2.20	.68	.04	1.95	.70
20	19–21	2.20	.13	5.12	1.47	.70	.04	1.63	.50
23	22–24	2.26	.15	4.50	1.13	.72	.05	1.43	.36
27	25–29	2.33	.16	4.10	.96	.74	.05	1.30	.29
33	30–36	2.40	.19	3.80	.72	.76	.06	1.20	.23
42	37–48	2.48	.22	3.52	.63	.79	.07	1.12	.20
58	49–72	2.60	.25	3.33	.54	.83	.08	1.06	.17
97	73–144	2.83	.28	3.14	.44	.90	.09	1.00	.14
290	145–300	2.92	.31	3.00	.38	.93	.10	.95	.12
	Rack	2.96	.34	2.96	.34	.94	.11	.94	.11

diametral pitch of the gear in order to produce the face radius. Hence for this gear the face radius becomes 2.40 ÷ 3 = 0.80 in. To the right of *2.40* in the next column is the distance *0.19*, which when divided by the diametral pitch gives the radial distance from the pitch circle to the circle of face centers. This face distance is always measured radially inward from the pitch circle and for this gear is 0.19 ÷ 3 = 0.06 in. To obtain the flank radius and flank distance of the gear: opposite *30*, in the columns headed "Flanks" are found *3.80* and *0.72*. These are now divided by 3 to give 1.27 in. and 0.24 in., which are the flank radius and distance respectively. The flank distance is measured radially outward from the pitch circle. Had the circular pitch been used instead of the diametral pitch, the last four columns would have been used, and the numbers there given would have been multiplied by the circular pitch to yield the required radii and distances.

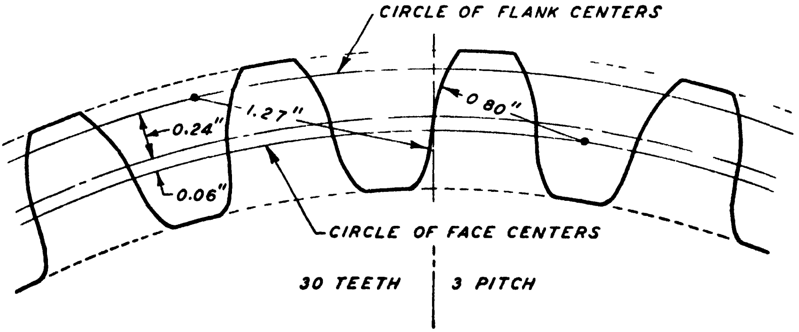


Fig. 122. Cycloidal Spur Gear Teeth

To proceed with the drawing, locate the circles of faces and flanks at their respective distances from, and concentric with, the pitch circle. Having now these circles of centers and the face and flank radii, the profiles of the teeth are drawn in exactly the same manner as by the method of Fig. 121.

An inspection of the odontograph table shows that the radius of the flanks of a 12-tooth gear is of infinite length; that is, the flanks are radial lines. This indicates that any gear having teeth designed from this table belongs to an interchangeable set of gears whose base is a 12-tooth pinion.

The Involute. When a straight line rolls upon a circle, the path of any point of the right line is a curve called the *Involute of the circle.*

In mechanism this circle is called the *Base Circle.* The involute may also be defined as the path which the end of an inextensible string takes as the string is unwound from a circle; for the string, as it is unwound and kept taut, takes the same successive tangent positions to the circle as those occupied by the rolling line.

The construction of the involute, shown in Fig. 123, is as follows: to the right of point *O,* lay off the small equal arcs *O-1*, *1-2*, *2-3*, etc. The latter are made small enough to permit their lengths to be approximated by their equal chordal lengths. Draw radial lines of the base circle to points *1*, *2*, *3*, etc., and erect a perpendicular at the end of each radial line as shown in the figure. These perpendiculars represent the successive positions of the rolling line (or the string) and are tangent to the base circle. Next lay off on the tangent at point *1*, a distance equal to the arc length *O1*, thus locating point *a* of our involute. In a similar manner points *b*, *c*, *d*, etc., are located by laying off distances on the tangents at *2*, *3*, *4*, etc., equal to arcs *O2*, *O3*, and *O4* respectively. In locating these points, arc *O1* is assumed equal in length to its chord *O1*. Hence arc *O2* is taken equal to two times the chordal length *O1*, arc *O3* is taken equal to three times the chordal

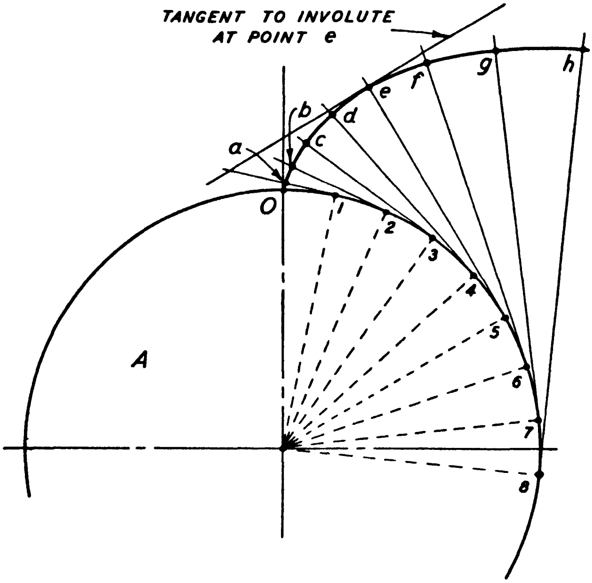


Fig. 123 Involute

length *O1*, etc. The points thus obtained are the successive positions of point *O* of the generating line, and a smooth curve drawn through them is the involute, *Oh*, of the base circle *A*.

It is evident from the construction that the involute must lie entirely outside its base circle. It should also be noted that the shape of an involute depends on the diameter of the base circle; hence for involute curves to be identical, they must have the same or equal base circles.

Every position of the line as it rolls upon the base circle is normal to the involute; that is, it is perpendicular to the tangent to the involute at that point of the line which is generating the involute. Thus line *5e* is perpendicular to the tangent to the involute at point *e*, as shown in the figure. Since the line as it rolls upon the base circle is always tangent to that circle, it follows that any normal to the involute is tangent to the base circle. This is a very important property of the involute.

The Involute as a Tooth Profile. In Fig. 124 a thin flat band (or cord) is wrapped partly around and attached to cylinders *A* and *B*, becoming tangent to the latter at points *a* and *b*, respectively. If cylinder *B* is caused to rotate in a counterclockwise direction about its center O_b, it will pull the band to the right and cause cylinder *A* to rotate in a clockwise direction about its center O_a. The action of the band with each cylinder must be one of pure rolling, due to the fact that each end of the band is attached to one of the cylinders and hence no slipping can occur. Thus Fig. 124 portrays the manner in which a straight line, *ab*, may be conceived to roll with two base circles, aa_1 and bb_1, simultaneously.

Any point of the line *ab*, such as *x*, while it moves in the direction of its line, will generate the involute *mn* of the base circle *A* and *rs* of the base circle *B* as the rolling action proceeds. The generating of these involutes is more easily seen by cutting *ab* at *x* and letting *ax* and *bx* roll upon (or be unwound from) circles *A* and *B* respectively, while the latter are held fixed. It is evident that the free end of *ax* will generate the involute *mn* of *A* and the free end of *bx* will generate the involute *rs* of *B*.

Now if these involutes are used for the profiles of a pair of teeth of *A* and *B* so that the latter can act as a direct contact pair, any point of contact between them will be a position of the common generating point x. Their common normal (which with an involute is always a position of the generating line) will therefore coincide with ab and cut the line of centers at the fixed point P. Such involute profiles will therefore conform to the law of gearing.

The constant angular velocity ratio that results from the use of the tooth profiles mn and rs, is inversely as the segments O_aP and O_bP of the line of centers. Circles which are described with these segments as radii and are concentric with the base circles have pure rolling and the same constant angular velocity ratio as that produced by these tooth profiles. Hence the action of a pair of involute gears is equivalent to the pure rolling of these circles (which are the pitch circles of the gears) with point P as the pitch point.

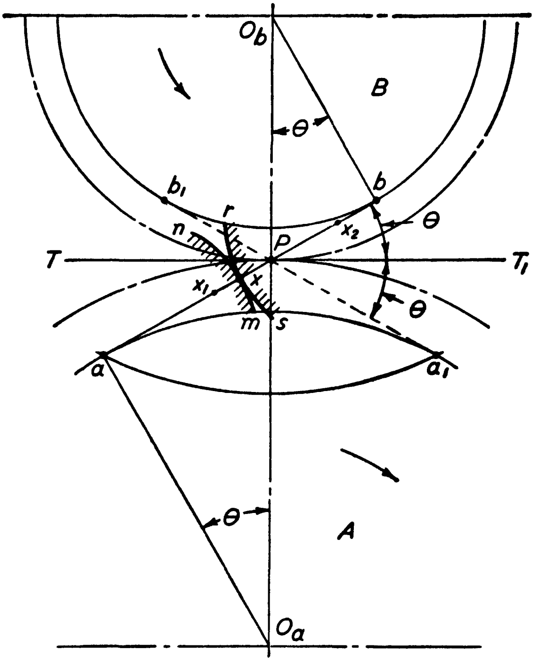


Fig 124

Let us suppose that the pair of teeth placed on A and B have acting profiles only on one side of each tooth as shown in the figure. With the direction of motion as given, A must be the driver; and the point of contact will move toward point b along the common normal.

If the direction of rotation is reversed, B must become the driver; and the point of contact will move toward point a along the same common normal or line of action, ab. Now let us suppose that in such a case B cannot act as the driver. It will then be necessary to provide each tooth of the pair with another profile placed on the opposite side of the tooth. (See Fig. 125.) These profiles will be generated having line a_1b_1, Fig. 124, roll upon the base circles of A and B as shown. The involutes generated by a_1b_1 will be the same as those generated by ab but they will be reversed in direction; hence each tooth will be symmetrical. With this second pair of contacting profiles, A, rotating in a counterclockwise direction can be the driver, and the point of contact will move toward b_1 along their common normal a_1b_1. From this discussion it is evident that, with symmetrical teeth, either A or B can act as the driver of the direct-contact pair, irrespective of the direction of rotation.

As previously stated, the path of the point of contact is along the common normal, which is a line tangent to both base circles. Consider the case in which A, the driver, is rotating clockwise. For contact to begin at a, the tooth on B must have an addendum circle passing through this first point of contact; therefore its addendum circle radius is O_ba. For contact to end at b, the tooth on A must have an addendum circle passing through this final point of contact; therefore its radius is O_ab. The point of contact cannot occur earlier than a nor later than b because an involute lies entirely outside its base circle. Therefore O_ab and O_ba are the maximum addendum circle radii, and ab is the maximum distance that the point of contact may travel. In any event, the addendum circle of the driver intersects the common normal at the final point of contact, and the addendum circle of the driven gear intersects the common normal at the initial point of contact. In other words, if the point of contact x is initially at x_1 and moves along ba to a final position x_2, then the radius of the addendum circle of the driver A is O_ax_2, while that of the follower B is O_bx_1.

In the preceding discussion it is evident that driving can continue only so long as the single pair of teeth are in contact. However, action between A and B may be made continuous by providing other similar pairs of teeth so designed and proportioned that they will work properly together.

Involute Spur Gear Teeth in Action. The base circles of Fig. 124, together with their involutes, *mn* and *rs*, have been used in the construction of the pair of involute spur gear teeth of Fig. 125. In the latter, gear *A* is the driver and gear *B* is the follower or driven gear. The addendum, pitch, base, and root circles of each gear are given by their names in the figure. To the left of the line of centers of the gears, the pair of teeth is shown in contact at point x_1, which is their initial point of contact. As motion proceeds, the point of contact moves along the common normal through the pitch point *P* to the final point of contact x_2, at which the pair of teeth is in the position shown in our figure to the right of the line of centers.

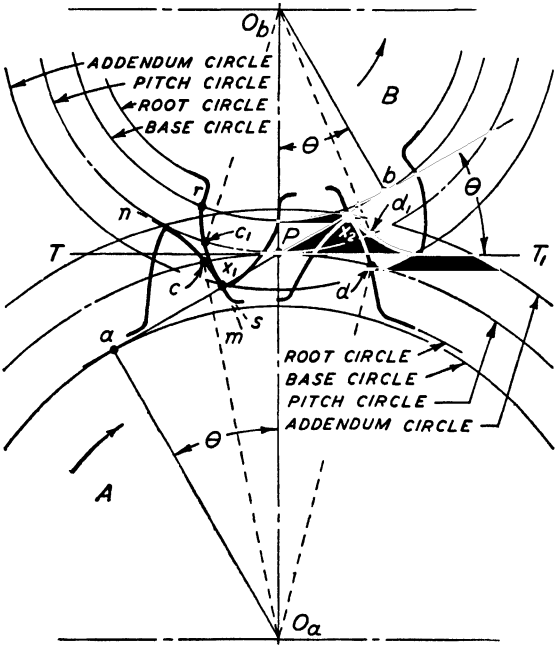


Fig. 125

When the mating teeth are beginning their action at x_1, the contacting profiles of the teeth cut their respective pitch circles at points *c* and c_1. These points become the point of contact when the teeth have moved into contact at the pitch point *P*, and by the time the teeth have reached x_2 and completed their contact, *c* and c_1 are at *d* and d_1 respectively. From this statement and the definitions that have been given in this chapter, it is evident that in Fig. 125,

cO_ad = the angle of action of A
$c_1O_bd_1$ = the angle of action of B
cO_aP = the angle of approach of A
c_1O_bP = the angle of approach of B
PO_ad = the angle of recess of A
PO_bd_1 = the angle of recess of B
cPd = the arc of action of A
c_1Pd_1 = the arc of action of B
cP = the arc of approach of A
c_1P = the arc of approach of B
Pd = the arc of recess of A
Pd_1 = the arc of recess of B

Since the action of these involute spur gears is equivalent to the pure rolling of their pitch circles, arc cP = arc c_1P, arc Pd = arc Pd_1, and arc cd = arc c_1d_1. For continuous action, gears must of course be so designed that when one pair of teeth is quitting contact, another pair must have already made contact or at least be ready to make contact. Therefore with the involute, as well as with the cycloidal, system of gears, the arc of action can never be less than the circular pitch.

The angle of obliquity or pressure angle, as stated heretofore, is the angle between the common normal and the common tangent to the pitch circles. In Fig. 125 it is the angle θ. This angle for mating involute gears is constant, since the path of the point of contact is along the generating line of their tooth profiles. An angle of obliquity of 14½ degrees is probably the most generally used in practice. The tooth proportions previously given in this chapter are suited to this angle and produce a so-called *Full-Depth* involute tooth. Other angles of obliquity in use at the present time are angles of 15 degrees, 20 degrees, and 22½ degrees.

Interference of Involute Teeth. In providing clearance for involute teeth, the root distance which is used will cause the root circle to be located outside the base circle with some gears, and inside the base circle with others. Both of these conditions are shown in Fig. 125. The root circle of gear A lies outside the base circle and thus permits the flanks of the teeth of A to be entirely involute in form. On the other hand, the root circle of gear B lies inside the base circle, and the flank cannot be entirely involute in form because the involute curve cannot extend inside its evolute, the base circle. In order to complete the flanks from the base circle to the root circle, radial lines are used as shown.

The tooth proportions arbitrarily used in the design of gear teeth may give to a gear a length of tooth which will cause its face to contact the radial portion of the flank of the mating tooth. Such contact is not between two involute curves which are generated by the same straight line, and therefore the teeth will not act properly together. This condition is called Interference. It is particularly in evidence when a small pinion meshes with a relatively large gear. Note that the teeth of Fig. 125 are so designed that they cannot interfere.

In order to avoid interference between teeth without making any change in their angle of obliquity and tooth proportions, the following modifications of the mating tooth profiles are resorted to:

(*a*) The radial portion of the flanks of the pinion teeth may be hollowed out or "undercut." This causes the teeth to be materially weakened.

(*b*) That portion of the face of the gear which tends to contact the radial portion of the flank of the pinion may be trimmed off or "relieved." This is the better of the two methods. It should be noted, however, that this method decreases the arc of action.

(It should be noted that a radial flank, when used on a cycloidal pinion, does not cause interference because it is a hypocycloid generated by the same rolling circle that generates the epicycloidal gear face that works with it.)

Drawing Involute Spur Gear Teeth. The method of drawing these teeth is presented in connection with Fig. 126 and is accomplished by means of graphically obtaining the base circle and generating the involute tooth profile thereof. The tooth proportions as previously given for cast gears, together with an angle of obliquity of 14½ degrees, will be adopted. It is assumed that the number of teeth and the circular pitch are given.

Calculate the pitch radius by multiplying the number of teeth by the circular pitch and dividing the product thus obtained by 2π. Locate the center of the gear, describe the pitch circle, and draw its vertical and horizontal center lines. Locate the pitch point P at the intersection of the pitch circle and the vertical center line, and through P draw TT_1 tangent to the pitch circle. Draw line hh_1 through P making an angle of 14½ degrees with TT_1. Drop a perpendicular from center O to line hh_1, intersecting the latter at g. Then Og is the radius of a circle to which hh_1 is tangent. Describe

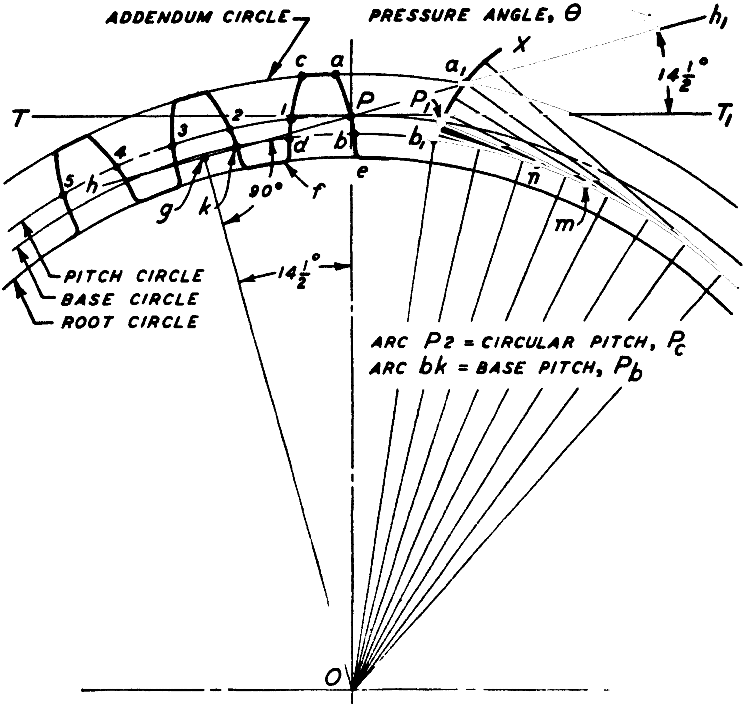


Fig. 126. Layout of Involute Teeth

a circle with O as a center and Og as a radius. This is the base circle of the gear. Assume the thickness of tooth = the width of space $= \frac{P_c}{2}$. Starting with P, lay off several arc lengths, each equal to $\frac{P_c}{2}$, obtaining points *1*, *2*, *3*, etc. Calculate the addendum and dedendum distances and describe the addendum and dedendum circles concentric with the pitch circle. At any point b_1, generate the involute b_1x of the base circle by the method of Fig. 123. The addendum circle cuts this involute at a_1; hence P_1a_1 is the involute face, and P_1b_1 is the involute portion of the flank of the tooth profile. In order to transfer this curve to the point on the pitch circle through which the profile passes, either a templet or the approximate arc method can be used. The latter, which is used in the figure, consists of finding, by trial, radii of circular arcs which will approximate P_1b_1 and P_1a_1. The circle of centers of these approximate arcs is the base circle of the involute. (With large gears the face and flank may generally be approximated by a single arc.) Center the compass at different points along the base circle until one is reached that will cause the compass to travel along P_1b_1. In the figure this is point *n*, so that nP_1 is the approximate flank radius. In a similar manner find point *m*, which gives mP_1 as the approximate face radius. Now keeping the compass centered along the base circle, and with radius nP_1, describe arcs such as *Pb* (from the pitch circle to the base circle) through points *P*, *2*, *4*, etc. Reversing the compass, describe arcs such as *1d* through points *1*, *3*, *5*, etc. Centering the compass again along the base circle with mP_1 as a radius, describe arcs such as *Pa* through points *P*, *2*, *4*, etc.; and reversing the compass draw arcs such as *1c* through points *1*, *3*, *5*, etc. Since the base circle lies outside the root or dedendum circle, complete the flanks by drawing radial lines such as *be* and *df* from the base circle to the root circle. The teeth are now completed by inserting the fillets.

(It should be noted that the teeth of Figs. 121 and 126 are for gears having the same number of teeth and circular pitch. A comparison of the two figures will clearly indicate the difference in the profiles of the two systems.)

Grant's Involute Odontograph Method. As in the case of cycloidal teeth, a method has been devised by Mr. George B. Grant whereby the teeth of involute gears can be drawn without actually generating the involute curves for the tooth profiles. This *Involute Odontograph* is given in Table II. It furnishes the face and flank radii to be used to approximate an involute profile when either the diametral pitch or circular pitch is known. As stated in the table, values taken from the diametral pitch columns must be divided by the diametral pitch of the gear, while those taken from the circular pitch columns must be multiplied by the circular pitch which is used. No so-called "distance" is given to locate the circles of centers because, as seen in the previous article, the base circle of the gear is always the circle of centers for both faces and flanks. This base circle is located by the odontograph at a distance of one-sixtieth of the pitch diameter or $\frac{D_p}{60}$ inside the pitch circle.

TABLE II—Grant's Involute Odontograph

Teeth	Divide by the Diametral Pitch		Multiply by the Circular Pitch	
	Face Radius	Flank Radius	Face Radius	Flank Radius
10	2.28	.69	.73	.22
11	2.40	.83	.76	.27
12	2.51	.96	.80	.31
13	2.62	1.09	.83	.34
14	2.72	1.22	.86	.39
15	2.82	1.34	.90	.43
16	2.92	1.46	.93	.47
17	3.02	1.58	.96	.50
18	3.12	1.69	.99	.54
19	3.22	1.79	1.03	.57
20	3.32	1.89	1.06	.60
21	3.41	1.98	1.09	.63
22	3.49	2.06	1.11	.66
23	3.57	2.15	1.13	.69
24	3.64	2.24	1.16	.71
25	3.71	2.33	1.18	.74
26	3.78	2.42	1.20	.77
27	3.85	2.50	1.23	.80
28	3.92	2.59	1.25	.82
29	3.99	2.67	1.27	.85
30	4.06	2.76	1.29	.88
31	4.13	2.85	1.31	.91
32	4.20	2.93	1.34	.93
33	4.27	3.01	1.36	.96
34	4.33	3.09	1.38	.99
35	4.39	3.16	1.39	1.01
36	4.45	3.23	1.41	1.03
37–40	4.20		1.34	
41–45	4.63		1.48	
46–51	5.06		1.61	
52–60	5.74		1.83	
61–70	6.52		2.07	
71–90	7.72		2.46	
91–120	9.78		3.11	
121–180	13.38		4.26	
181–360	21.62		6.88	

In order to see the application of this method, the teeth of a 24-tooth, 3-pitch involute spur gear will be drawn, see Fig. 127. To obtain the radius R_p, of the pitch circle, from formula (47),

$$P_d D_p = t$$

or
$$D_p = \frac{t}{P_d}$$

Since
$$R_p = \frac{D_p}{2}$$

$$R_p = \frac{t}{2P_d}$$

In this example, $t = 24$ teeth and $P_d = 3$. Therefore

$$R_p = \frac{24}{2 \times 3} = 4 \text{ in.}$$

From tooth proportions based on the diametral pitch,

$$\text{the addendum} = \frac{1 \text{ in.}}{P_d} = \frac{1}{3} \text{ in.}$$

and
$$\text{the dedendum} = \frac{1.157 \text{ in.}}{P_d} = \frac{1.157}{3} = 0.39 \text{ in.}$$

The radius of the addendum circle

$$= R_p + \text{the addendum}$$
$$= 4 + \tfrac{1}{3} = 4\tfrac{1}{3} \text{ in.}$$

The radius of the root circle

$$= R_p - \text{the dedendum}$$
$$= 4 - \tfrac{1}{3} = 3\tfrac{2}{3} \text{ in.}$$

From formula (49), the circular pitch,

$$P_c = \frac{\pi}{P_d} = \frac{\pi}{3} = 1.047 \text{ in.}$$

Also from the tooth proportions,

$$\text{thickness of tooth} = \text{width of space}$$
$$= \frac{P_c}{2} = 0.524 \text{ in.}$$

The distance from the pitch circle to the base circle

$$= \frac{D_p}{60} = \frac{2 \times 4}{60} = 0.13 \text{ in.}$$

Therefore the radius of the base circle

$$= R_p - \frac{D_p}{60} = 4 - 0.13 = 3.87 \text{ in.}$$

Having secured the above dimensions, the drawing can be made. Referring to Fig. 127, locate the center of the gear and describe the pitch, base, addendum, and root circles. On either side of the pitch point, *P*, lay off the points on the pitch circle through which the profiles are to pass. To secure the radii to be used in drawing these profiles, locate the number of teeth, *24*, in the first column of Table II. Since the diametral pitch is being used, locate, under the latter and to the right of *24*, the face and flank radii, *3.64* and *2.24* inches, respectively. The face radius is then

$$3.64 \div 3 = 1.21 \text{ in.}$$

and the flank radius is

$$2.24 \div 3 = 0.75 \text{ in.}$$

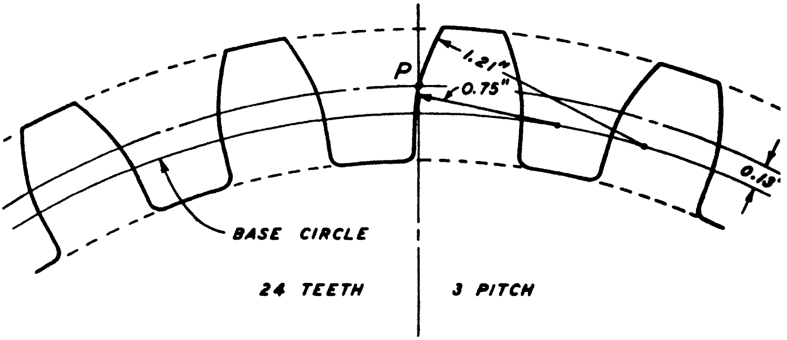


Fig. 127

With the compass set to the face radius and centered along the base circle, describe arcs from the pitch circle to the addendum circle, being careful that the arcs on the right sides of the teeth are the reverse in direction of those on the left sides. Noting that the center of the flank arc is on the base circle and on the same side of the profile as the corresponding face arc, set the compass to the flank radius and describe arcs from the pitch circle to the base circle. Since the base circle falls outside the root circle, extend each flank from the base circle to the root circle by a radial line. Complete the teeth by inserting their fillets.

Stub-Tooth Gears. Many involute gears employ teeth which are shorter than those of standard proportions, the so-called "full-depth" teeth. Due to this characteristic, they are known as *Stub-Tooth Gears*. The teeth of such gears have a shorter addendum and dedendum than the standard involute tooth and a pressure angle of 20 degrees instead of 14½ degrees.

The *Fellows Stub Tooth** was developed by the Fellows Gear Shaper Co. of Springfield, Vt. It is based on two diametral pitches which are written $^3/_4$, $^4/_5$, $^5/_7$, etc. The first figure of each of these combinations of pitches is the actual diametral pitch of the gear, in that it is used in determining the pitch diameter, circular pitch, thickness of tooth, and the number of teeth. The second figure then is used to determine only the addendum and the dedendum.

The American Gear Manufacturers' Association has adopted a stub tooth which is based on a single diametral pitch, the relative tooth proportions being as follows:

$$\text{Pressure angle} = 20 \text{ degrees}$$

$$\text{Addendum} = \frac{0.80 \text{ in.}}{P_d}$$

$$\text{Dedendum} = \frac{1.00 \text{ in.}}{P_d}$$

$$\text{Working depth} = \frac{1.60 \text{ in.}}{P_d}$$

$$\text{Whole depth} = \frac{1.80 \text{ in.}}{P_d}$$

$$\text{Clearance} = \frac{0.20 \text{ in.}}{P_d}$$

Angular Velocity Ratio of Spur Gears. Referring to the spur gears A and B of Fig. 111 (*b*), let

t_a = the number of teeth on gear A

t_b = the number of teeth on gear B

D_{pa} = the pitch diameter of gear A

D_{pb} = the pitch diameter of gear B

R_{pa} = the pitch radius of gear A

R_{pb} = the pitch radius of gear B

From statements previously made, the angular velocity ratio of A to B,

$$\frac{V_{aa}}{V_{ab}} = \frac{R_{pb}}{R_{pa}} \tag{53}$$

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*For further material on this type of tooth the student is referred to a booklet entitled, *The Stub-Tooth Gear* which is published by the Fellows Gear Shaper Co , Springfield, Vt

Multiplying both numerator and denominator of the second member by 2,

$$\frac{V_{aa}}{V_{ab}}=\frac{2R_{pb}}{2R_{pa}}$$

or

$$\frac{V_{aa}}{V_{ab}}=\frac{D_{pb}}{D_{pa}} \tag{54}$$

Multiplying both numerator and denominator of the second member by π,

$$\frac{V_{aa}}{V_{ab}}=\frac{\pi D_{pb}}{\pi D_{pa}} \tag{55}$$

From the figure it is evident that the pitch circumferences πD_{pb} and πD_{pa} are directly proportional to the numbers of teeth, or

$$\frac{\pi D_{pb}}{\pi D_{pa}}=\frac{t_b}{t_a}$$

Therefore from formula (55)

$$\frac{V_{aa}}{V_{ab}}=\frac{t_b}{t_a} \tag{56}$$

Since the r.p.m. ratio $\frac{N_a}{N_b}$ is equal to the angular velocity ratio, $\frac{N_a}{N_b}$ may replace $\frac{V_{aa}}{V_{ab}}$ in any of these formulas.

From the foregoing formulas it is seen that the angular velocity (or r.p.m.) ratio of a pair of spur gears is constant and equal to the inverse ratio of their pitch radii, their pitch diameters, their pitch circumferences, or their numbers of teeth.

Example 1. Two meshing spur gears, A and B, have 24 teeth and 42 teeth respectively. What is the r.p.m. of B when A is making 350 turns per minute?

Solution. Here $N_a=350$ r.p.m., $t_a=24$ teeth and $t_b=42$ teeth. Applying formula (56)

$$\frac{N_a}{N_b}=\frac{t_b}{t_a}$$

and evaluating therein

$$\frac{350}{N_b}=\frac{42}{24}$$

Multiplying both members of this equation by 24 N_b,

$$42\ N_b=24\times350$$

Dividing by 42

$$N_b = \frac{24 \times 350}{42} = 200 \text{ r.p.m.} \quad \textit{Ans.}$$

Example 2. A 9-inch spur gear, while making 175 revolutions per minute, causes its follower to make 75 revolutions per minute. What is the size (pitch diameter) of the follower?

Solution. Here $D_{pa} = 9$ in., $N_a = 175$ r.p.m., and $N_b = 75$ r.p.m. Applying formula (54)

$$\frac{D_{pb}}{D_{pa}} = \frac{N_a}{N_b}$$

and evaluating

$$\frac{D_{pb}}{9} = \frac{175}{75}$$

or

$$D_{pb} = \frac{175}{75} \times 9 = 21 \text{ in.} \quad \textit{Ans.}$$

Interchangeable Involute Spur Gears. All members of an interchangeable set of involute spur gears must be designed to have:

1. the same pitch
2. the same angle of obliquity
3. the same tooth proportions

The three requirements combine to permit the teeth to fit or mesh properly, to give continuous action, and to obey the law of gearing. Various constant angular velocity ratios can then be obtained from a set by mating pairs of gears of different numbers of teeth.

Alteration of the Distance between Centers of Involute Gears. It is usually difficult in practice to set or maintain a pair of mating gears at the exact center distance for which they were designed. It is essential with cycloidal gears that their center distance be kept always equal to the sum of their pitch radii, otherwise they will not transmit motion at a constant angular velocity ratio. However involute gears may be used with varying center distances and their angular velocity ratio will remain unchanged and constant; hence this is a very important property of the involute system. It will now be explained in conjunction with Fig. 128.

In this figure, two involute gears, A and B, are initially centered at O_a and O_b respectively. $O_a a$ or R_{ba} is the radius of the base circle of A, while $O_b b$ or R_{bb} is the radius of the base circle of B. The pitch circles of the gears are not shown, but their point of tangency is evidently P, so that their pitch radii are O_aP and O_bP. In the similar right triangles, O_aPa and O_bPb,

$$\frac{O_bb}{O_aa}=\frac{R_{bb}}{R_{ba}}=\frac{O_bP}{O_aP}\text{(corresponding sides of similar triangles are proportional)} \qquad (57)$$

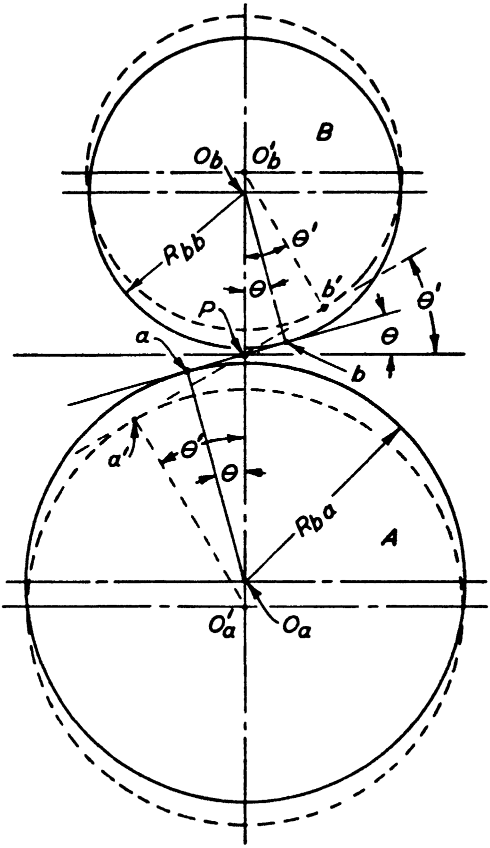


Fig. 128

Formula (57) states that the base (circle) radii of a pair of involute gears are directly proportional to the pitch radii.

But from formula (53)

$$\frac{V_{aa}}{V_{ab}}=\frac{O_bP}{O_aP}$$

therefore

$$\frac{V_{aa}}{V_{ab}}=\frac{R_{bb}}{R_{ba}} \qquad (58)$$

Formula (58) states that the angular velocity ratio of a pair of involute spur gears is equal to the inverse ratio of the base radii.

Now let the gears be moved away from each other so that O_a and O_b take the new positions O_a' and O_b', their base circles take the dotted positions as shown, and the center distance is increased from O_aO_b to $O_a'O_b'$. The common tangent to the base circles takes the new position or becomes $a'b'$, so that the angle of obliquity is increased from θ to θ'. Now $a'b'$, in rolling on the base circles, will generate the same profiles that the gears had in their initial positions, for the radii of the base circles have remained unchanged in length and involutes of the same or equal circles are identical. Therefore the involute profiles in these new positions are still conjugate, so that from formula (58)

$$\frac{V_{aa}}{V_{ab}}=\frac{O_b'b'}{O_a'a'}=\frac{O_bb}{O_aa}=\frac{R_{bb}}{R_{ba}}$$

and the constant angular velocity ratio has remained unchanged.

The pitch point P may or may not remain in the same position while the center distance is changed. In either case, however, the pitch radii are altered; so that, from a theoretical standpoint, the gears have different pairs of tangent pitch circles for each new position. However, the ratio of the pitch radii remains constant so that, as the center distance varies, the angular velocity ratio is always equal to the inverse ratio of the radii of the original pitch circles upon which the design of the gears was based.

There are of course limits to the variations in the center distance. The minimum center distance will place the mating gears in a position where their backlash is equal to zero, while the maximum is that at which the gears are so situated that but one pair of teeth are in contact at any instant.

Fig. 129 shows an involute gear in mesh simultaneously with two gears. Since each is a 10-pitch gear having 40 teeth, the pitch diameter upon which they were designed is 40÷10=4 inches. If the two mating pairs were so placed that their designed pitch circles were tangent, the center distance for each pair would be the sum of their pitch radii, or 4 inches. However one pair has a center distance which is greater than 4 inches while the center distance of the other pair is less than 4 inches. This clearly illustrates the possibility of varying the center distance when involute gears are in use.

Base Pitch of an Involute Gear. It has been shown that a change in the center distance of a pair of involute gears produces, theoretically, a change in the pitch radius of each gear. This in turn affects the circular pitch because it is equal to the circumference of the pitch circle divided by the number of teeth.

The base radius of an involute gear is unaffected by a change in the center distance. Evidently then, if the circumference of the base circle is divided by the number of teeth, the resulting quotient, called the *Base Pitch*, P_b (see Fig. 126), is a constant arc length on the base circle. It may also be defined as the distance from a point on one tooth to the corresponding point on the next tooth, the distance being taken on the base circle.

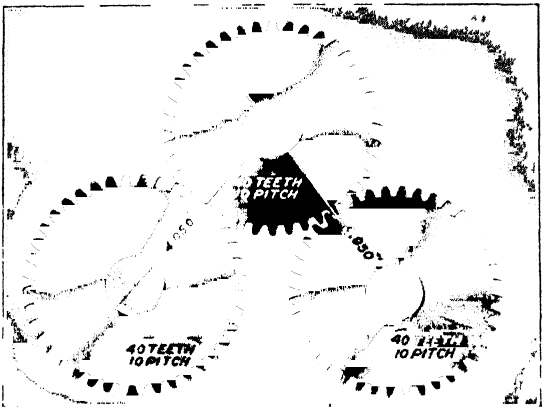


Fig 129 Variation of Center Distance with Involute Gears
Courtesy of Fellows Gear Shaper Co., Springfield, Vt

Referring to the initial position of the gears of Fig. 128, from the right triangle O_aPa,

$$\frac{O_a a}{O_a P} = \cos\theta$$

or

$$\frac{R_{ba}}{R_{pa}} = \cos\theta$$

from which

$$R_{ba} = R_{pa}\cos\theta \qquad (59)$$

Multiplying both members of formula (59) by $\frac{2\pi}{t_a}$,

$$\frac{2\pi R_{ba}}{t_a} = \frac{2\pi R_{pa}\cos\theta}{t_a}$$

But $$\frac{2\pi R_{ba}}{t_a} = P_b, \text{ by definition}$$

and $$\frac{2\pi R_{pa}}{t_a} = P_c, \text{ by definition}$$

therefore $$P_b = P_c \cos \theta \qquad (60)$$

It is evident from formula (60) that the base pitch P_b will be the same for each gear of a pair of mating gears because both gears at any given center distance must have the same circular pitch P_c and the same angle of obliquity or pressure angle, θ.

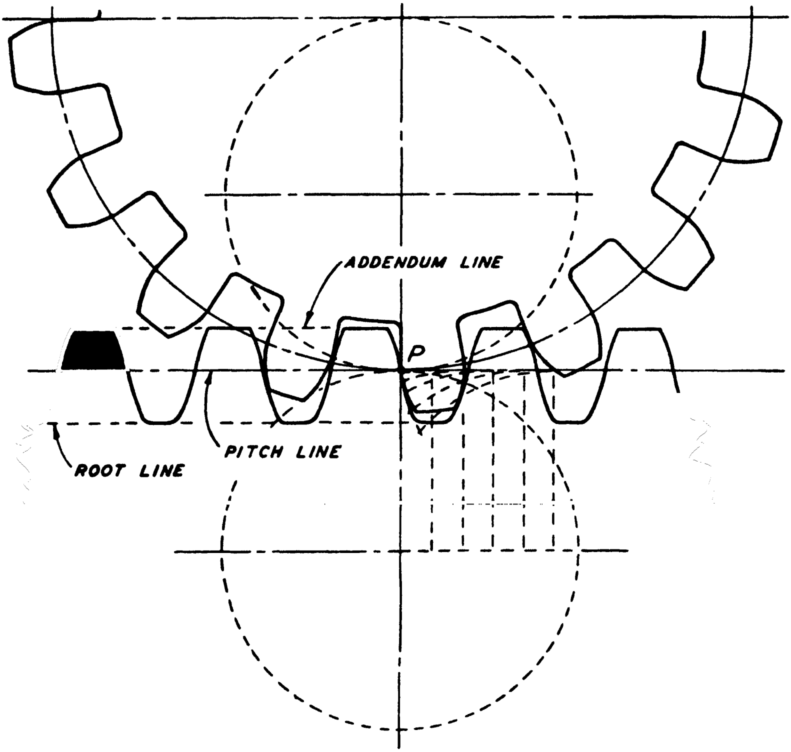


Fig. 130. Cycloidal Rack and Pinion

Rack and Pinion. A gear which has a straight line for its pitch circle is known as a *Rack,* while the gear with which it meshes is known as a *Pinion.* The latter is a typical spur gear and its pitch circle is tangent to the pitch line of the rack at the pitch point P. The addendum and root circles of the rack become straight lines drawn parallel to its pitch line. A rack and pinion may employ either the cycloidal or the involute system for their tooth profiles.

A cycloidal rack and pinion are shown in mesh in Fig. 130. They may be made members of an interchangeable set by using the generating circle of the set for their faces and flanks. In this manner any gear of the set will work properly with the rack. The generating circle rolls on both the upper and lower sides of the pitch line of the rack, giving the latter cycloidal faces as well as cycloidal flanks, so that cycloidal faces of the rack contact the hypocycloidal flanks of the pinion, and the cycloidal flanks of the rack contact the epicycloidal faces of the pinion. The teeth may be drawn by either generating their profiles as indicated in the figure or by using the odontograph. In case the latter method is used, the teeth of the 16-tooth pinion will not have the radial flanks shown in the figure, for the odontograph gives radial flanks to a 12-tooth pinion.

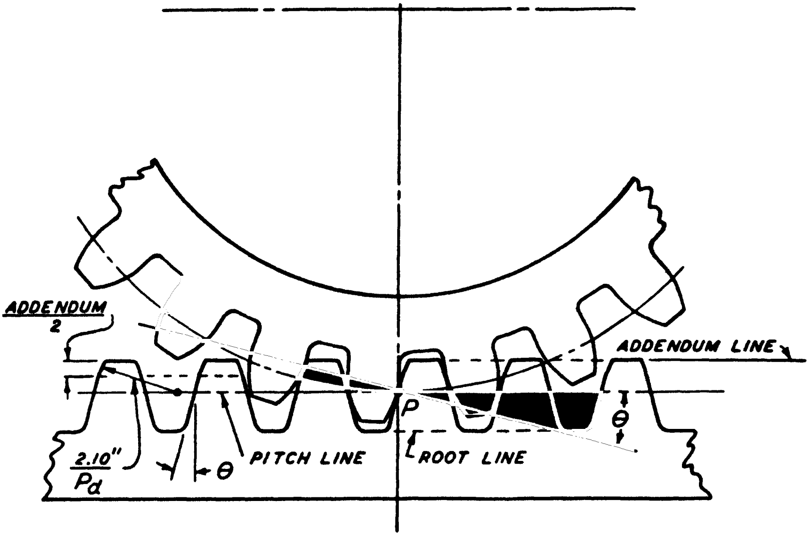


Fig 131 Involute Rack and Pinion

An involute rack and pinion are shown in Fig. 131. Since the pitch line of the rack is a straight line, the base circle (which is necessarily parallel to it) becomes a straight line. The involute of the latter is a straight line perpendicular to the line of action or obliquity. Hence involute rack teeth have straight sides which intersect the pitch line at an angle θ, the angle of obliquity of the gears. Where involute racks are to mesh with pinions which have a relatively small number of teeth, the upper end of each face of the rack is rounded off, through a distance equal to one-half the addendum, by an arc whose radius is equal to 2.10 inches divided by the diametral pitch. This is done in order to avoid interference of the teeth.

It is evident that the motion of a rack must be reciprocating, with the direction of rotation of the pinion changing periodically.

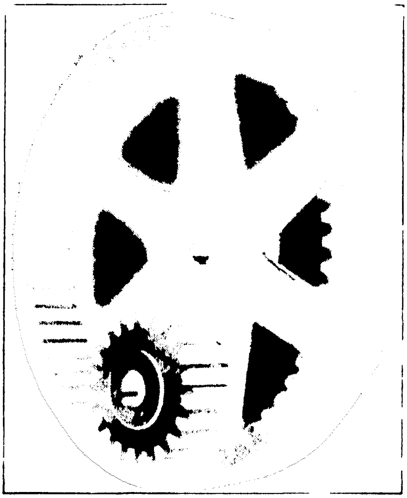


Fig 132 Annular Gear and Pinion
Courtesy of Stephens-Adamson Mfg Co, Aurora, Ill

Annular Gears. It will be remembered from the preceding chapter that a pair of rolling cylinders located on parallel shafts can be tangent externally or internally. (See Figs. 97, 107, and 108.) In the case of external contact or tangency, the cylinders rotate in opposite directions and the distance between their axes is equal to the sum of their radii; while with internal tangency the cylinders rotate in the same direction, and their center distance is equal to the difference of their radii.

Mating spur gears not only employ cylinders with external tangency for their pitch surfaces, but those with internal tangency as well. The latter condition is illustrated in Fig. 132. Here a typical small spur gear or pinion is in mesh with a larger gear whose teeth are on the inside of the rim and therefore point toward the center of the gear. The larger gear is called an *Annular Gear* (or *Wheel*) and the pair is spoken of as an *Annular Gear and Pinion.*

The tooth profiles of such a pair of gears may be either cycloidal or involute in form. Since the addendum circle is always that circle which limits the tops of the teeth, it is the innermost circle of the annular gear. The addendum distance is therefore measured radially inward while the dedendum distance is measured radially outward from the pitch circle. It is interesting to note that when the cycloidal system is employed with an annular gear and pinion, the hypocycloidal faces of the gear contact hypocycloidal flanks of the pinion, while the epicycloidal flanks of the gear contact epicycloidal faces of the pinion. A peculiar form of interference known as "fouling" is apt to be present between the teeth of annular gears unless they are properly designed.

Annular gears are often referred to as *Internal* gears to distinguish them from *External* or regular spur gears. Their internal contact not only permits of the same direction of rotation, but also presents several distinct advantages over external gears, among which may be noted:

1. Compactness of design due to the small center distance
2. Safety in operation; the annular gear as seen in Fig. 132 is really a guard to the pair of mating gears, so that a protective cover is not so necessary as with external gears
3. Greater efficiency and smoothness in operation
4. Less wear over the tooth surfaces

Comparison of the Cycloidal and Involute Systems. Although the cycloidal tooth form was at first used almost exclusively in the manufacture of gears, it was rapidly superseded by the involute tooth form, once the latter was introduced and found superior in most ways; so that today the involute tooth form practically dominates the field of gear manufacture. However the cycloidal tooth is still used in certain special cases, and the cycloidal curve is used in combination with the involute curve to form the tooth profile of the basic rack of the so-called "14½-degree Composite System" which has been approved by the A.S.A. (American Standards Association). Therefore a knowledge of the theory of the cycloidal tooth form is still essential and is of great aid in understanding the action of gear teeth in general.

The advantages of the involute system which have placed it in its favored position in gear-tooth design are as follows:

1. Variation in center distance due to setting of gears, type of service or wear at the bearings causes no change in the angular velocity ratio.
2. The path of contact or the common normal is a straight line and the angle of obliquity is constant. These characteristics are conducive to smoothness in running, less wear, and a higher efficiency.
3. Form cutters or generating cutters for involute teeth are more easily and cheaply manufactured than those used for cycloidal teeth, due to the simpler form of the involute.
4. The teeth of an interchangeable set of involute gears of a given pitch may be cut with as few as eight form cutters, while twenty-four such cutters are required with the cycloidal system.

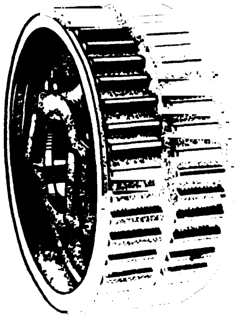


Fig 133 Stepped Gear

Stepped Gears. Suppose that a certain number, say three, of thin or narrow spur gears which are identical in every respect are bolted together in such a manner that the teeth of the second gear are advanced one-third of the pitch ahead of the teeth of the first, and the teeth of the third gear are advanced a like amount ahead of those of the second, the pitch elements of the three gears lying in the same cylindrical surface. Such an assembly, if constructed as a single unit, is called a *Stepped Gear.* One of these gears having two steps is illustrated in Fig. 133. In the design of some stepped gears the ends of the teeth are connected by extensions of the rim which project outward along the flanks of the teeth. Teeth so connected are stronger and more rigid and are said to be shrouded.

In general, a pair of stepped gears behave the same as spur gears, in that their angular velocity ratio is inversely as the pitch radii, the pitch diameters, or the numbers of teeth; and the contacting sections or steps of mating teeth have an action which is identical to that of the teeth of spur gears, being a combination of rolling and sliding in the plane of rotation.

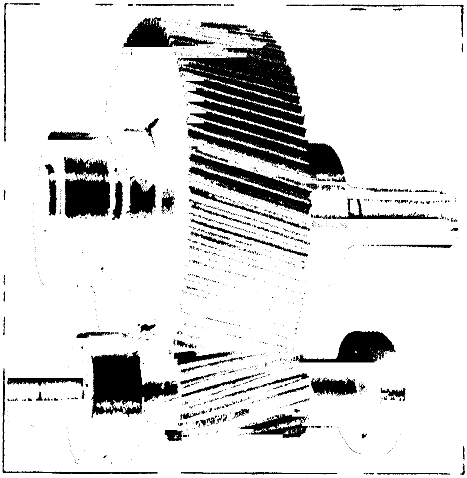


Fig. 134 Helical Spur Gear and Pinion in Mesh
Courtesy of Foote Bros. Gear and Machine Corp., Chicago, Ill.

It will be remembered that spur gear teeth have pure rolling only at the instant they are in contact at the pitch point (or at their common pitch element). By creating steps in the teeth, it is evident that such contact at the pitch point will occur more frequently, hence the action in stepped gears is quieter and smoother than that in straight spur gears. It will be noted in the next article that this smoothness of action is present to an even greater degree in helical spur gears. Since the latter are also more cheaply made, stepped gears are seldom used.

Helical Spur or Twisted Gears. If the number of steps on a stepped gear is increased indefinitely, the steps will become infinitely thin, resulting in a gear with continuous teeth whose elements are curved or twisted uniformly into a regular helical form with respect to the axis of the gear. Such a gear is known as a *Helical Spur* or *Twisted Gear*. A gear and pinion of this type are shown in mesh in Fig. 134, in which it may be noted that mating helical spur gears must have helices of opposite hand. In the figure, the gear and pinion employ right-hand and left-hand helices respectively. The directions of the helices become apparent when the axes of the gears are placed in a vertical position.

These gears have line contact between their teeth which is of such a nature that a point of contact is always afforded somewhere along the common contact element of their parallel pitch cylinders. For this reason, helical spur gears run much more smoothly and quietly than straight spur gears and hence are particularly adaptable to high speeds. It should be evident that the angular velocity ratio of a pair of mating helical spur gears is inversely proportional to their pitch radii, pitch diameters, and numbers of teeth. Hence with these gears formulas (53), (54), and (56) apply.

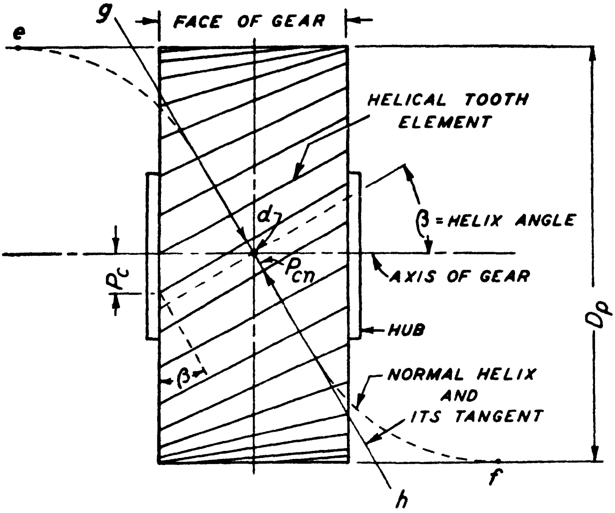


Fig 135 Pitch Cylinder of a Helical Gear

Helix Angle and Normal Pitch of a Helical Gear. Several new terms must be introduced for use with all helical gears, whether the helical spur gears already mentioned or the helical gears to be introduced shortly. These terms were not needed in the discussion of a regular or straight-tooth spur gear, primarily because its tooth elements are straight lines parallel to the axis of the gear, so that the normal or right section of a spur gear tooth lies in the plane of rotation and is therefore represented by the end of the tooth. Since a normal tooth section of these gears with helical elements does not lie in the plane of rotation, other pitches must be introduced. These are the normal circular pitch and the normal diametral pitch upon which a normal section is designed.

The pitch cylinder of a helical gear is shown in Fig. 135. Upon it are located those pitch elements which are cut from the pitch surface by the corresponding surfaces of adjacent teeth. These helical pitch elements, represented by straight lines in the figure, make the angle β with the elements of the pitch cylinder. Angle β is the so-called "Helix Angle" of the gear. (See angle β of Fig. 13, Chapter II.) The distance P_c, measured along the pitch circle in a plane perpendicular to the axis (the plane of rotation) is the circular pitch. It corresponds to the circular pitch used with spur gears. The diametral pitch, P_d, is of course equal to $\frac{\pi}{P_c}$.

The line *ef* is a helix which intersects a pitch element at point *d* in such a manner that their tangents at that point are perpendicular to each other. Therefore *ef* is a *Normal Helix* and a section of tooth taken along *ef* is a *Normal Section.* The distance between corresponding elements measured along a normal helix is the *Normal Circular Pitch,* P_{cn}. The *Normal Diametral Pitch,* P_{dn}, is evidently equal to $\frac{\pi}{P_{cn}}$.

In order to establish a relationship between the circular pitch and the normal circular pitch of a helical gear, as well as one between their diametral and normal diametral pitches, from the figure,

$$\frac{P_{cn}}{P_c} = \cos\beta$$

or

$$P_{cn} = P_c \cos\beta \tag{61}$$

Since $P_{cn} = \frac{\pi}{P_{dn}}$, and $P_c = \frac{\pi}{P_d}$, by substituting in formula (**61**),

$$\frac{\pi}{P_{dn}} = \frac{\pi}{P_d}\cos\beta$$

Dividing by π and multiplying by $P_{dn}P_d$,

$$P_d = P_{dn}\cos\beta$$

or

$$P_{dn} = \frac{P_d}{\cos\beta} \tag{62}$$

It is interesting to note that if the helix angle β is made equal to zero degrees, the helical elements become straight lines parallel to the axis of rotation, the gear becomes a straight-tooth spur, and, from formulas (61) and (62), $P_c=P_{cn}$ and $P_d=P_{dn}$. Hence normal pitches are not mentioned in spur gearing.

End Thrust in Helical Gears. In any direct-contact mechanism the force exerted by the driver upon the follower is directed along the common normal; hence in helical spur gears this normal force or pressure between the teeth is located with respect to the tooth element and axis of the gear (the axis of rotation) in the manner illustrated in Fig. 136. Here the vector F represents the tooth pressure. Since it is normal to the tooth element, it necessarily makes an angle equal to the helix angle β with the plane of rotation and therefore sets up an end thrust equal to the axial component F_1. This end thrust tends to move the gear or its shaft along its axis and must be resisted by the installation of a thrust bearing. As previously stated, the helices of

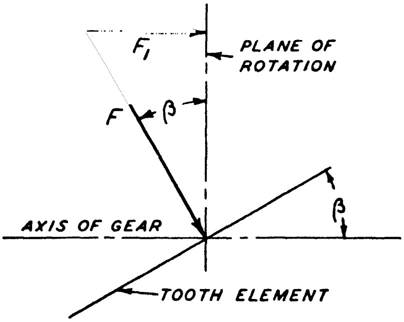


Fig 136 End Thrust

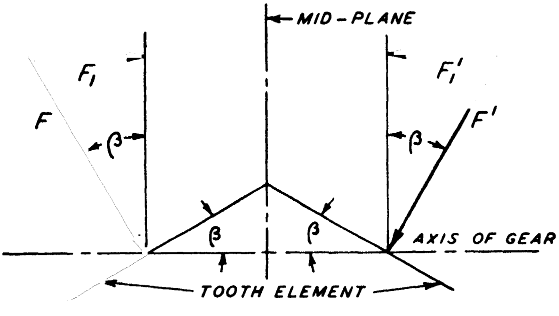


Fig 137

a pair of mating helical spur gears are of opposite hand; therefore the end thrust along the shaft of the driver is in the opposite direction to that along the shaft of the follower.

Herringbone Gears. Now a helical spur gear will be considered in which the elements of the teeth on one side of the mid-plane are opposite in hand to those on the other side, as shown in Fig. 137. In such a case the normal tooth pressure is equally distributed to the two sides of the gear so that $F=F'$ and, as shown in the figure, the axial component F_1 of F is equal in magnitude, but opposite in direction, to the axial component F_1' of F'. The axial components therefore neutralize each other and no end thrust exists. Gears of this type are called *Double Helical Spur* or *Herringbone Gears.* The teeth of these gears may be *Continuous* as shown in Fig. 138 or *Staggered* as in Fig. 139. When the teeth are staggered, herringbone gears are commonly called Wuest gears, being given the name of the Swiss engineer who invented them.

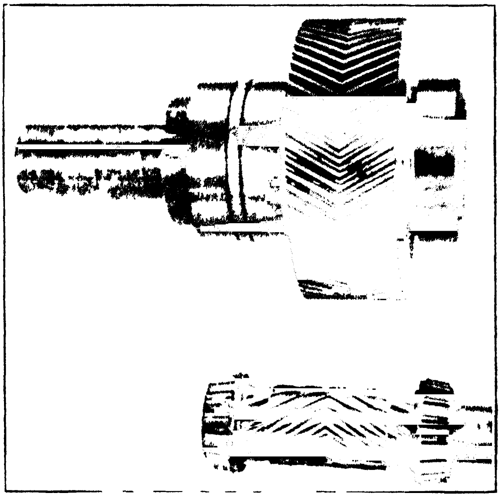


Fig 138 Herringbone Gear and Pinion
Courtesy of Link-Belt Co., Chicago, Ill

Helical Gears. As already noted, gears with helical tooth elements, when used to connect parallel shafts, are called *Helical Spur* or *Twisted Spur Gears,* because their action is similar to that of straight spur gears. Other gears with helical tooth elements are used to connect shafts which are non-parallel and non-intersecting.

These are known as *Helical Gears* and are shown in Fig. 140. They must not be confused with helical spur gears. The pitch surfaces of helical gears are right circular cylinders which, in the case of mating gears, are tangent at a point, due to the relative position of the axes.

Two mating helical gears transmit motion between their shafts at a constant angular velocity ratio which is inversely as their numbers of teeth. Therefore formula (56) applies with this type as well as all other types of gears. However, with helical gears the angular velocities (or r.p.m.'s) are not inversely as either the pitch radii or the pitch diameters. Instead, it can be proved that the angular velocity ratio of a pair of mating helical gears, A and B, is given by the following formula:

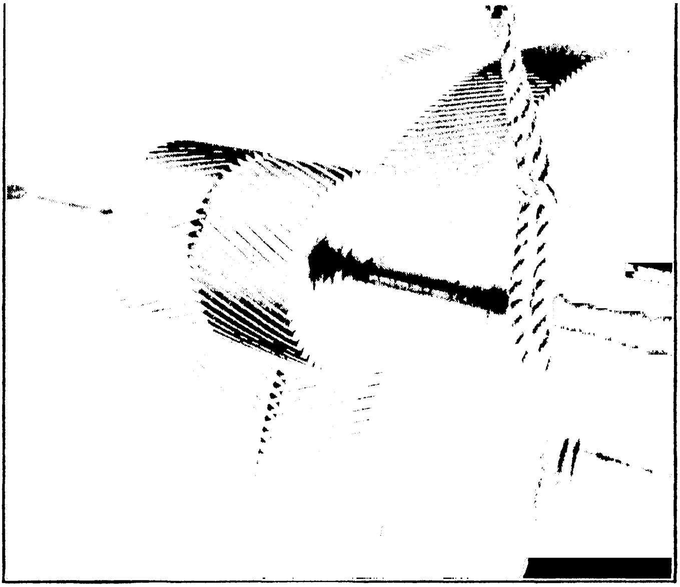


Fig 139. Staggered-Tooth Herringbone Gears

Courtesy of The Falk Corporation, Milwaukee, Wisconsin

$$\frac{V_{aa}}{V_{ab}}=\frac{N_a}{N_b}=\frac{D_{pb}\cos\beta_b}{D_{pa}\cos\beta_a}=\frac{R_{pb}\cos\beta_b}{R_{pa}\cos\beta_a} \qquad (63)$$

in which

β_a = helix angle of A

β_b = helix angle of B

and the other notation is as previously given.

The angle between the axes of a pair of helical gears is generally equal to 90 degrees, although it may take other values. If this angle is called by the name θ,

$$\beta_a + \beta_b = \theta \tag{64}$$

In other words, the sum of the helix angles of a pair of helical gears is equal to the angle between their axes.*

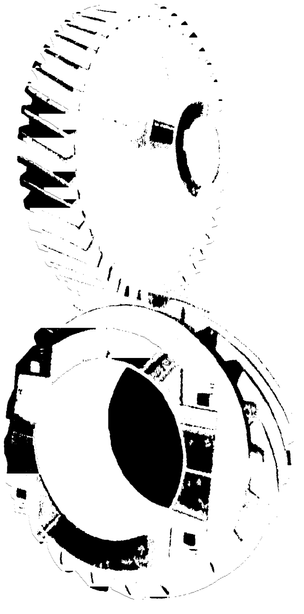


Fig 140 Helical Gears in Mesh
Courtesy of Foote Bros Gear and Machine Corp, Chicago, Ill

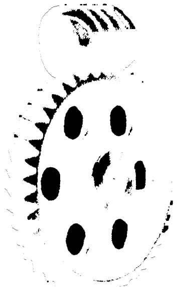


Fig 141 Worm and Worm-Wheel
Courtesy of Boston Gear Works, Inc, North Quincy Mass

Comparison of Helical Spur and Helical Gears. The characteristics of both helical spur and helical gears are presented in order, not only to define each type, but to expose, through comparison of their respective characteristics, the radical difference that prevails between the two types.

The characteristics of a pair of mating helical spur gears may be listed as follows:

1. They are used to transmit motion at a constant angular velocity ratio between parallel shafts.

---

*In exceptional cases which seldom if ever occur in practice, the difference rather than the sum of the helix angles is equal to the angle θ In such an event the teeth of the mating gears become opposite in hand instead of the same hand, as stated in the next article

2. Their pitch surfaces are right circular cylinders which contact each other along a line, since their axes are parallel.

3. Driver and follower must have the same helix angle.

4. Their tooth elements must be of opposite hand.

5. They necessarily have the same normal circular pitch and hence the same normal diametral pitch.

6. Their circular pitches are equal as are also their diametral pitches. This is evident from preceding statements *3* and *5*, and from formulas (**61**) and (**62**).

7. Their teeth give line contact.

8. The action of the mating teeth is a combination of rolling and sliding in the plane of rotation. There is no sliding along the tooth elements.

The characteristics of a pair of mating helical gears may be listed as follows:

1. They are used to transmit motion at a constant angular velocity ratio between shafts whose axes are non-parallel and non-intersecting.

2. Their pitch surfaces are right circular cylinders which contact each other at a single point due to the fact that the axes are non-parallel and non-intersecting.

3. The sum of the helix angles of driver and follower is equal to the angle between their axes. See formula (**64**). Their helix angles may or may not be equal to each other. More often they are not equal.

4. Their tooth elements are of the same hand.

5. They have, of course, the same normal pitches.

6. Their pitches (circular as well as diametral) are not equal except in the case where β_a is equal to β_b. This becomes evident from an inspection of formulas (**61**) and (**62**).

7. Their teeth afford point contact. They do not give line contact. Therefore these gears are not desirable when the power to be transmitted is relatively high.

8. Mating teeth have the customary rolling and sliding action which, however, is combined with another sliding action in the direction of, or along, the tooth elements.

Worm and Worm-Wheel. (See Fig. 141.) This pair of toothed wheels may be considered as a special case of helical gears. Like the latter, they are used to connect non-intersecting shafts which practically always are at right angles to each other. The worm is a typical helical gear and generally acts as the driver. The worm-gear or wheel is not a helical gear.

The construction of a worm is such that each tooth is wrapped around it in a helical manner, so that the worm resembles a threaded bolt. For this reason, the teeth of a worm are generally referred to as "threads." Hence the single-thread worm of Fig. 142 (*a*) is a worm

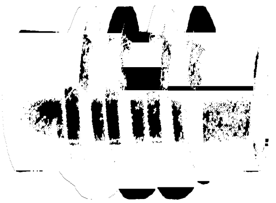


Fig 142(*a*) Single-Thread Worm

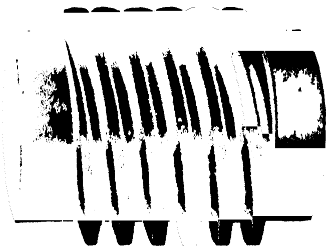


Fig 142(*b*) Double-Thread Worm

Fig 142(*c*) Quadruple-Thread Worm

Figs 142(a), (b) and (c), Courtesy of Foote Bros Gear and Machine Corp , Chicago, Ill.

which has one tooth, the double-thread worm of Fig. 142 (*b*) is a worm having two teeth, while the quadruple-thread worm of Fig. 142 (*c*) is one having four teeth. Worm gears of more modern design than those just described are shown in Fig. 142 (d).

As shown in Fig. 143, there are three general types of worm-wheels. The concave-face type, Fig. 143 (*c*), is much more efficient than the straight-face types of Fig. 143 (*a*) and (*b*). The teeth of the former fit partly around the worm and conform to the contour of the teeth on the worm, which permits the load and wear to be distributed over a larger area. Hence the concave type is used in all cases where a considerable amount of power is to be transmitted. Several worm-wheels of late design are illustrated in Fig. 143 (d).

The angular velocity ratio of driver to follower in worm-gear drives is inversely as the numbers of teeth; therefore formula (56) applies. Since the number of teeth on a worm is the same as its number of threads, it is a relatively small number; in fact, a single-thread worm is a gear having only one tooth. It is therefore evident that high velocity ratios can be obtained with this type of drive.

Example 1. A single-thread worm drives a worm-wheel whose number of teeth is 90. When the worm is making 1200 revolutions per minute, what is the rotative speed of the worm-wheel?

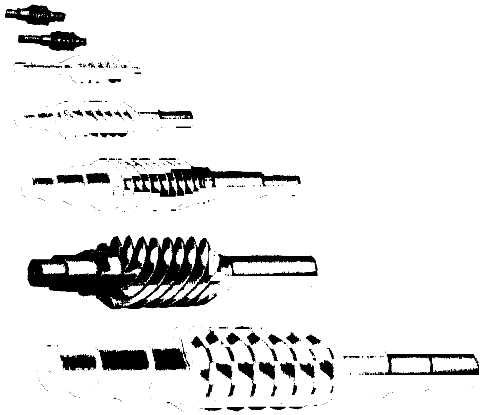


Fig. 142(*d*). Group of Worm Gears
Courtesy of Foote Bros. Gear and Machine Corp., Chicago, Ill.

Solution. Let $t_a = 1$ tooth, then $t_b = 90$ teeth, and $N_a = 1200$ r.p.m. From formula (56)

$$\frac{N_b}{N_a} = \frac{t_a}{t_b}$$

Evaluating

$$\frac{N_b}{1200} = \frac{1}{90}$$

$$N_b = \frac{1200}{90} = 13\tfrac{1}{3} \text{ r.p.m.} \quad \textit{Ans.}$$

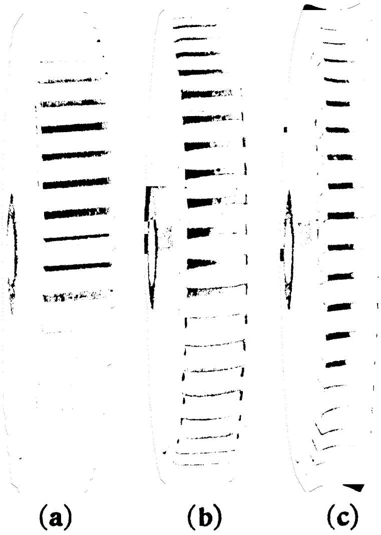
Example 2. A triple-thread worm drives an 80-tooth worm-wheel. If the latter is rotating at 30 r.p.m., what is the speed of the worm?

Solution. Here $t_a = 3$ teeth, $t_b = 80$ teeth and $N_b = 30$ r.p.m. Applying formula (56)

$$\frac{N_a}{N_b} = \frac{t_b}{t_a}$$

and evaluating $$\frac{N_a}{30}=\frac{80}{3}$$

$$N_a=\frac{80}{3}\times 30=800 \text{ r.p.m.} \quad \textit{Ans.}$$


(a) (b) (c)

Figs 143(a) (b) and (c) Worm-Wheels

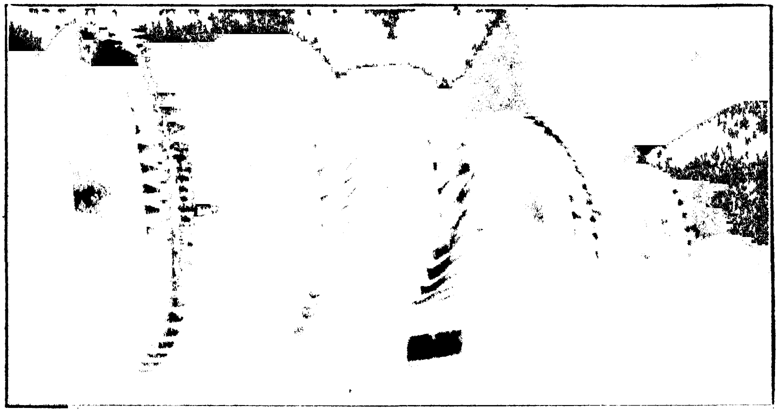


Fig 143(d) Worm-Wheels of Late Design

Figs 143(a), (b), (c), and (d), Courtesy of Foote Bros Gear and Machine Corp, Chicago, Ill

Bevel Gears. Toothed wheels which are used to connect two shafts whose axes intersect are called *Bevel Gears.* (See Fig. 144.)

They provide positive driving and a constant angular velocity ratio between the shafts. The angular velocity ratio of the driving and driven bevel gears is inversely as the numbers of teeth and also inversely as the pitch diameters, or pitch radii, as is the case with spur gears. Therefore formulas (53), (54) and (56) may be used to secure the velocity ratio of mating bevel gears.

As previously stated in this text, bevel gears result theoretically from the alteration of the surfaces of rolling right circular cones. This alteration is effected in order to obtain positive driving. These conical surfaces then become the pitch surfaces of the gears. Hence the teeth of bevel gears follow along a conical surface. This causes the teeth to get progressively smaller from one end to the other. All elements of the teeth, if extended, will pass through the apex of the pitch cone and, with a pair of mating bevel gears, the apex of each pitch cone must be located at the intersection of the axes of the gears. (See Fig. 146.) It is due to this fact that bevel gears are made in pairs only, and not in interchangeable sets. The size of a bevel gear is given by the pitch diameter of its outer, or largest, pitch circle.

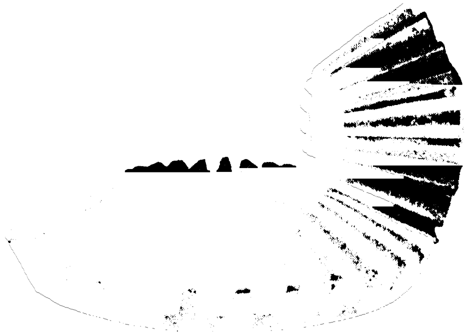


Fig 144 Bevel Gears
Courtesy of Link-Belt Co., Chicago, Ill

It has become customary to call a pair of bevel gears by the name of *Miter Gears* when their shafts make an angle of 90 degrees with each other and have the same rotative speed. It is evident from this statement that a pair of miter gears must have the same number of teeth and therefore be of the same size. Bevel gears whose axes intersect at any angle other than 90 degrees are known as *Angular Gears.* See Fig. 145.

The involute system is generally employed for the tooth profiles of bevel gears. The cycloidal system finds some application in cast bevel gears.

Bevel Gear Terms. Most terms used with bevel gears are the same as those used with the other types of gears and have the same interpretation in all cases. The definitions of this article are given to clarify some terms previously used and to introduce others which are peculiar to bevel gears. Reference should be made to Fig. 146 which is a typical sectional view of a pair of these gears.

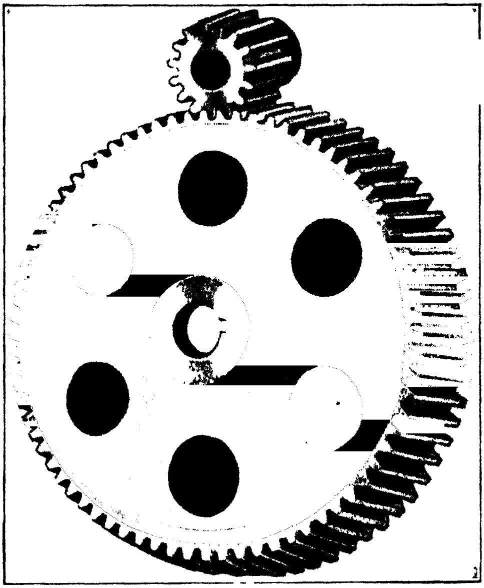


Fig 145 Angular Gear and Pinion
Courtesy of Foote Bros Gear and Machine Corp, Chicago, Ill

The *Pitch Cones* are *oab* and *obc*. Note that their bases *ab* and *bc* are the pitch circles of the gears and are tangent at point *b*. The cones are tangent along their common element *ob*.

The addendum, pitch and root circles of a bevel gear are those located at its outer end. Hence when the diameter of one of these circles is given, it is understood to be so located. Notice the addendum circle diameter, *O.D.* and the pitch circle diameter D_p, of Fig. 146.

The *Cone Distance* is the slant height of the pitch cone of a bevel gear. It is therefore the true length of an element of the pitch cone.

Two mating bevel gears have the same cone distance, because their pitch circles (the bases of the pitch cones) are tangent and their apexes are located at the same point.

The *Mounting Distance* is the distance from the center line of one gear to the end of the hub of the mating gear.

The *Backing, X,* is the distance from the (outer) pitch circle of a bevel gear to the end of its own hub.

The *Back Cone* of a bevel gear is a cone whose elements are perpendicular to the corresponding elements of the pitch cone at the outer end or pitch circle of the gear. Therefore both the back cone and the pitch cone have the pitch circle of the gear for a base. The apexes of the back cones in Fig. 146 are k and h respectively. The back cone is also called the *Normal Cone.*

The *Face, F,* of the gear is the length of its tooth measured parallel to the pitch element.

The pitch, either the diametral or circular, of a bevel gear always refers to that of the large end of the tooth or gear.

The apex, o, of the pitch cone is called the *Cone Center.* It is the intersection of the axes of the mating gears.

The angle between the addendum element of a tooth and the axis of the gear is called the *Face Angle.* In Fig. 146, coh is the face angle of the pinion.

The angle between the pitch element of a tooth and the axis of the gear is called the *Pitch Angle.* In Fig. 146, θ_1 (angle aoh) and θ_2 (angle bok) are the pitch angles of the pinion and gear respectively. The sum of the pitch angles of a pair of mating gears is equal to the angle at which the axes of the gears intersect. In other words

$$\theta_1+\theta_2=\phi \qquad (65)$$

The angle between the root element of a tooth and the axis of the gear is called the *Root Angle* or *Cutting Angle.* The cutting angle of the pinion in Fig. 146 is the angle goh.

Layout of Pitch and Back Cones of Mating Bevel Gears. In order to demonstrate this layout, obtain the pitch and back cones of the mating gears A and B of Fig. 146. Their axes intersect at 90 degrees and

$$t_a=16 \text{ teeth}$$
$$t_b=24 \text{ teeth}$$
$$P_d=3$$

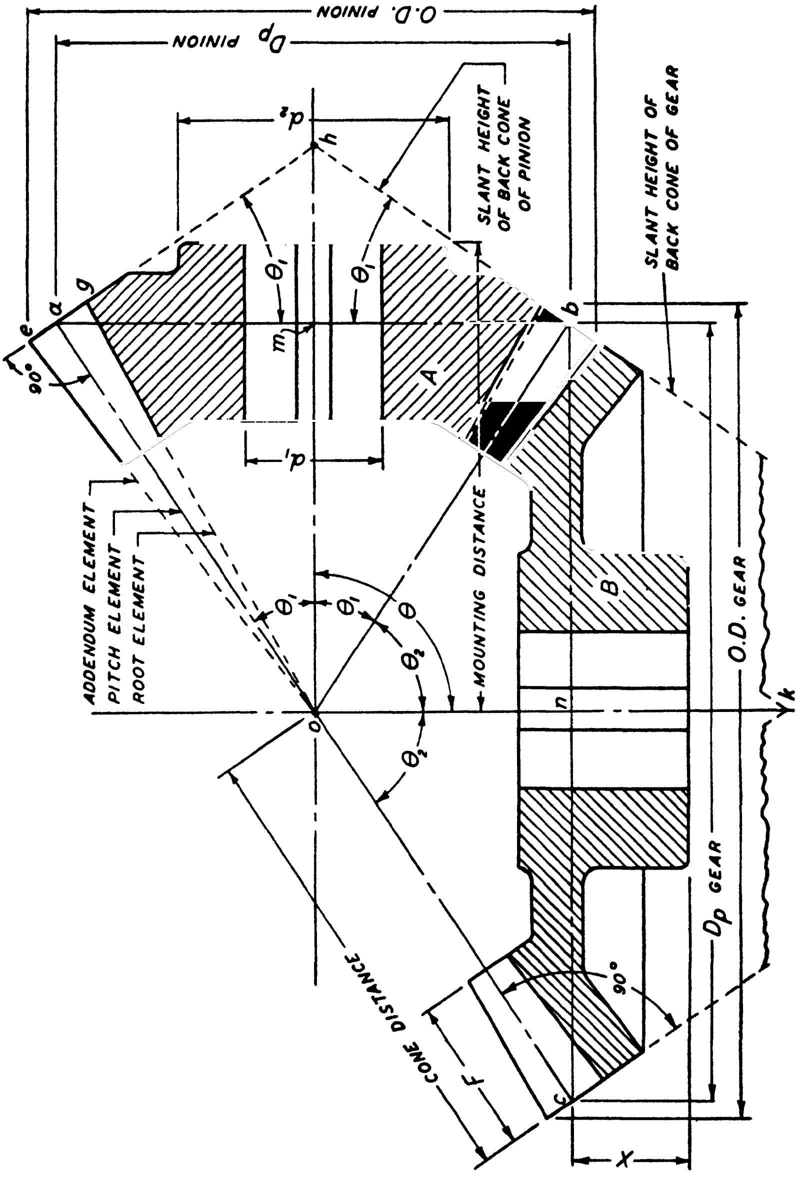


Fig. 146. Sectional View of Bevel Gear and Pinion

It will be necessary at the beginning to compute the pitch diameters and pitch radii of the gears. Since the pitch diameter is equal to the number of teeth divided by the diametral pitch, the pitch diameter of the pinion A is

$$D_{pa}=\frac{16}{3}=5\tfrac{1}{3}\text{ in.}$$

and the pitch diameter of the gear B,

$$D_{pb}=\frac{24}{3}=8\text{ in.}$$

The corresponding radii are therefore

$$R_{pa}=\frac{5\frac{1}{3}}{2}=2.667\text{ in.}$$

$$R_{pb}=\frac{8}{2}=4\text{ in.}$$

Referring to Fig. 146, draw the axes of A and B at 90 degrees to each other and intersecting at point o. On the axis of A, locate point m so that*

$$om=R_{pb}=4\text{ in.}$$

On the axis of B, locate point n in such a manner that

$$on=R_{pa}=2.6667\text{ in.}$$

Through m draw a line parallel to the axis of B and through n draw a line parallel to the axis of A. These lines intersect at point b, completing the parallelogram $ombn$, the diagonal of which is the common contact element of the pitch cones. Draw this diagonal ob. Extend line bm to point a making $ma=mb$. Extend line bn to point c making $nc=nb$. Draw lines oa and oc. Then oab and obc are the pitch cones of A and B respectively. ab is the pitch diameter of pinion A and the projection of its pitch circle. bc is the pitch diameter of the gear and the projection of the gear's pitch circle.

Draw lines ah and bh perpendicular to oa and ob respectively. These lines intersect on the axis of A at point h, and hab is the back cone of the pinion. In a similar manner, bk and ck are drawn perpendicular to ob and oc respectively, intersecting at point k on the axis of gear b; and kbc is the back cone of gear B.

---

*Note. For any angle of intersection, ϕ, it can be proved that $om=\frac{R_{pb}}{\sin\phi}$ and $on=\frac{R_{pa}}{\sin\phi}$. Here $\phi=90°$, so that $\sin\phi=1$. Therefore $om=R_{pb}$ and $on=R_{pa}$

Tredgold's Approximation. The outer ends of the teeth of a bevel gear theoretically lie in the surface of a sphere whose radius is the cone distance. In order to obtain the true form of the teeth, it would be necessary to develop the spherical surface in which their outer ends lie. This however is extremely difficult; hence a simplified method known as *Tredgold's Approximation* is employed. This method assumes with little error that the outer ends of the teeth lie in the surface of the back cone of the gear, and that the profiles of the teeth are for that reason the same as those of an *Equivalent Spur Gear* whose pitch radius is the slant height (see Fig. 146) of the back cone and whose pitch is of course the pitch of the bevel gear itself.

Formative Radius and Formative Number of Teeth of a Bevel Gear. The slant height of the back cone of a bevel gear, which becomes by Tredgold's method the pitch radius of the equivalent spur gear, is also known as the *Formative Radius* of the bevel gear. For gear A of Fig. 146 (or Fig. 147) it is represented by ha or hb. The number of teeth of the equivalent spur gear is likewise called the *Formative Number of Teeth* of the bevel gear. It is evident that the formative number of teeth is greater than the actual number of teeth of a bevel gear, for its formative radius is always larger than its pitch radius.

In order to obtain a relationship between these two numbers of teeth, let

t = the actual number of teeth
T = the formative number of teeth
R_p = the pitch radius
R_f = the formative radius
θ_1 = the pitch angle

Referring to Fig. 147, which is a drawing of the pinion of Fig. 146, note that the sides of angle mah are perpendicular to the sides of angle aoh, therefore

$$\angle mah = \angle aoh = \theta_1$$

and

$$\cos\theta_1 = \frac{ma}{ha} = \frac{R_p}{R_f} \qquad \text{(a)}$$

or

$$R_f = \frac{R_p}{\cos\theta_1} \qquad \text{(66)}$$

Since the number of teeth is equal to the circumference of the pitch circle divided by the circular pitch,

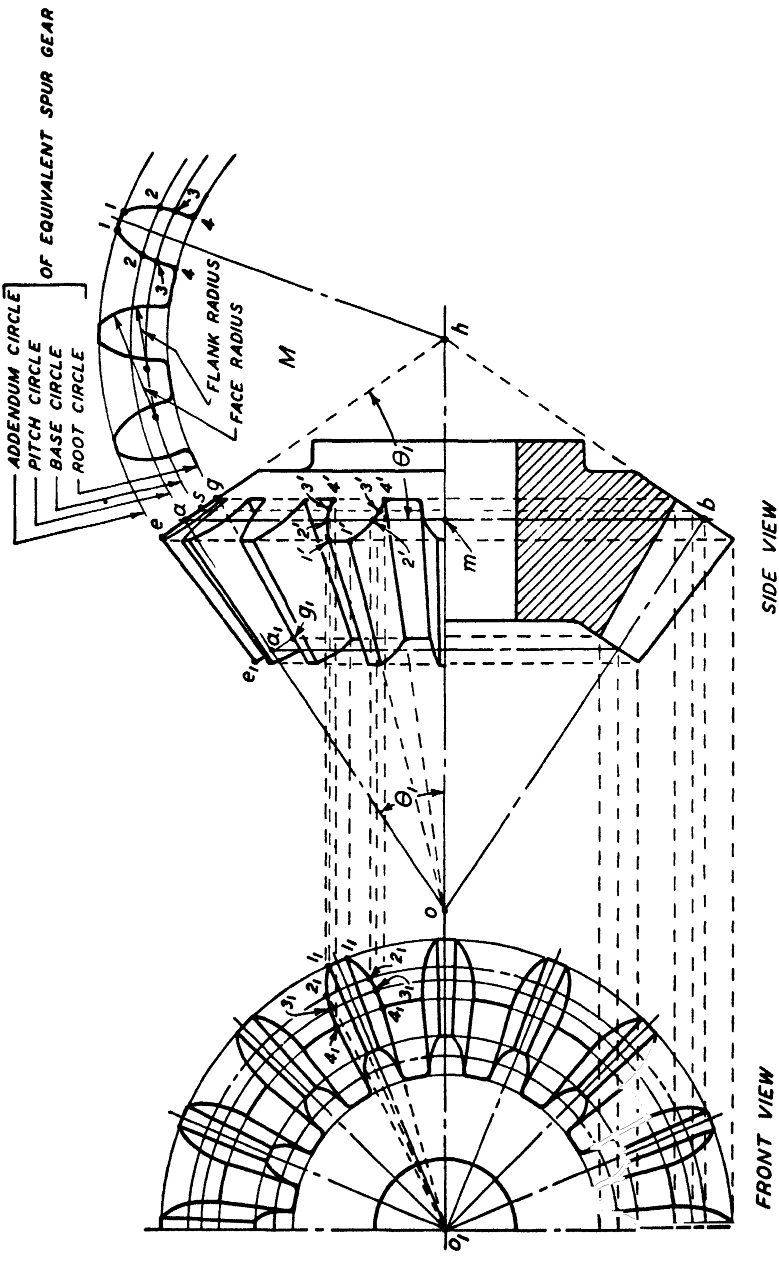


Fig 147 Tredgold's Method

$$t=\frac{2\pi\times(\textit{distance ma})}{P_c}=\frac{2\pi R_p}{P_c} \qquad \text{(b)}$$

and
$$T=\frac{2\pi\times(\textit{distance ha})}{P_c}=\frac{2\pi R_f}{P_c} \qquad \text{(c)}$$

From steps (*b*) and (*c*)

$$\frac{t}{T}=\frac{\dfrac{2\pi R_p}{P_c}}{\dfrac{2\pi R_f}{P_c}}$$

$$\frac{t}{T}=\frac{2\pi R_p}{P_c}\times\frac{P_c}{2\pi R_f}=\frac{R_p}{R_f} \qquad \text{(d)}$$

Therefore from steps (*a*) and (*d*)

$$\frac{t}{T}=\cos\theta_1$$

$$t=T\cos\theta_1$$

$$T=\frac{t}{\cos\theta_1} \qquad \text{(67)}$$

Formula (**67**) states that the formative number of teeth of a bevel gear is equal to its actual number of teeth divided by the cosine of its pitch angle.

Drawing Bevel Gear Teeth. Fig. 147 illustrates how to draw the teeth of the 16-tooth, 3-pitch involute gear *A*, of Fig. 146. It will be necessary at first to apply Tredgold's Method in order to obtain the true shape of the teeth as shown at *M* in Fig. 147. It will then be possible through the principles of orthographic projection to draw the projections of the teeth.

Referring to Fig. 147, lay out the pitch cone *oab* as in the preceding figure. Draw lines *ah* and *bh* perpendicular to *oa* and *ob* respectively. These lines will intersect at point *h* on the axis of the gear. Point *h* is the apex of the back cone *hab*, and *ha* is the formative radius of the bevel gear. Therefore, *ha* is the pitch radius of the equivalent spur gear.

In order to draw the true tooth forms at *M*, the following data must be obtained. From Fig. 146

$$\theta_1=\tan^{-1}\frac{ma}{om}=\tan^{-1}\frac{R_{pa}}{R_{pb}}$$

Since $R_{pa}=2.6667$ in. and $R_{pb}=4$ in.

$$\theta_1=\tan^{-1}\frac{2.6667}{4}=\tan^{-1}0.6667-$$

Therefore $\theta_1 = 33° 41'$

and $\cos\theta_1 = 0.8321$

From formula (66)

$$R_f = \frac{R_p}{\cos\theta_1}$$

$$= \frac{2.6667}{0.8321} = 3.2 \text{ in.}$$

Applying formula (67)

$$T = \frac{t}{\cos\theta_1}$$

and evaluating therein

$$T = \frac{16}{0.8321} = 19.2, \text{ say } 19 \text{ teeth}$$

The value of T may also be obtained by multiplying the pitch diameter of the equivalent spur by the diametral pitch. Thus from formula (47)

$$T = P_d\ (2R_f)$$

$$= 3 \times 2 \times 3.2 = 19.2, \text{ say } 19 \text{ teeth}$$

Addendum $ae = \frac{1 \text{ in.}}{P_d} = \frac{1}{3}$ in.

Root $ag = \frac{1.157 \text{ in.}}{P_d} = \frac{1.157}{3} = 0.386$ in.

Distance $as = \frac{1}{60}$ $\left\{\begin{array}{l}\text{of the pitch diameter of the equivalent spur}\\ \text{gear}\end{array}\right.$

$$= \frac{2R_f}{60} = \frac{R_f}{30} = \frac{3.2}{30} = 0.11 \text{ in.}$$

Circular pitch $P_c = \frac{\pi}{P_d}$ [See formula (49)]

$$= \frac{3.1416}{3} = 1.0472 \text{ in.}$$

Thickness of tooth = the width of space

$$= \frac{P_c}{2} = \frac{1.0472}{2} = 0.5236 \text{ in.}$$

From the involute odontograph, Table II, for 19 teeth and a diametral pitch of 3,

Face radius $= \frac{3.22}{3} = 1.07$ in.

Flank radius $= \frac{1.79}{3} = 0.60$ in.

To continue the drawing, on line *ha* lay off the distances, *ae*, *as*, and *ag*. With *h* as a center and radii *he*, *ha*, *hs*, and *hg*, describe the addendum, pitch, base, and root circles of the equivalent spur gear. Starting at any convenient point on the pitch circle of the equivalent spur, lay off several successive points at equal distances of $\frac{P_c}{2}$ from each other. Using the above face and flank radii and the base circle as a circle of centers, through these points describe the faces such as *1-2*; the flanks, *2-3*; and complete the flanks from the base to the root circle with a radial line; so that the true tooth forms are those of a spur gear which has T teeth and the pitch of the bevel gear as its pitch.

In order to draw the projections of the teeth on the front and side views, locate point a_1, on oa, so that a_1a is the face of the gear. This is generally taken equal to $\frac{oa}{3}$. From points e and g, draw lines which converge at the cone center o and intersect, at e_1 and g_1 respectively, a line drawn through a_1 perpendicular to oa. Then ee_1, aa_1, and gg_1 are addendum, pitch, and root elements respectively. e_1, a_1, and g_1 are points on the addendum, pitch, and root circles of the small end, just as e, a, and g are points on the corresponding circles of the large end of the gear. The dotted vertical lines through these points representing the projections of these circles on the side view are now drawn.

Next, extend the horizontal center line of the front view and locate, at some place thereon, point o_1, which is the center of the front view and the projection of axis of the gear. Through o_1 draw a vertical center line. Project across from points e, a, s, and g to the vertical center line and describe circles through the points thus obtained. These circles are respectively the projections of the addendum, pitch, base, and root circles of the large end of the gear. In a similar manner project across from e_1, a_1, and g_1, to obtain the addendum, pitch, and root circles of the small end.

Lay off on the front view a radial center line for each tooth. With the dividers, take the tooth thicknesses *1-1*, *2-2*, *3-3*, and *4-4* and transfer them to each of these tooth center lines as shown by 1_1-1_1, 2_1-2_1, 3_1-3_1, and 4_1-4_1. Draw a smooth curve (generally the arc of a circle) through each series of points, forming in this manner the projections of the profiles of the teeth. Elements of the teeth can now be drawn toward o_1 from 1_1, 2_1, and 4_1 (3_1 can be neglected as 3 points on the small end are sufficient) intersecting, in order, the addendum, pitch, and root circles of the small end at points which, if connected by a smooth curve, will give the projection of the tooth at the small end.

Project now from such points as 1_1, 2_1, 3_1, and 4_1 to $1'$, $2'$, $3'$, and $4'$ of the side view. Draw through the latter the converging elements of the teeth as shown. Connect all series of points by smooth curves as shown to complete the projections of the teeth.

Other Types of Gears. This chapter has introduced one or more types of gearing for each of the several relative positions in which a pair of shafts that are to be geared together may be placed. It has presented plain (straight toothed) spur, annular, helical spur and herringbone gears for connecting parallel shafts, plain bevel gears for connecting shafts with intersecting axes, and helical and worm gears for those shafts which are neither parallel nor intersecting. In addition to these, two more types of gearing frequently met in practice should be mentioned.

Fig 148 Spiral Bevel Gear and Pinion
Courtesy of Gleason Works, Rochester, New York

Spiral Bevel Gears shown in Fig. 148 are used to connect shafts with intersecting axes. They may be conceived to be formed by twisting the teeth of plain bevel gears in much the same manner as helical spur gears result from twisting the teeth of plain spur gears. However, unlike helical spur teeth, the curvature of spiral bevel teeth is generally of a circular nature rather than helical. This type of gear possesses the same advantages over the plain bevel gear that the helical spur has over the plain spur; namely, smoother operation, less noise, less impact, and higher speeds.

Hypoid Gears shown in Fig. 149, although similar in general appearance to the spiral bevel, differ in the relative position of the shafts which they connect; for the axis of the pinion does not intersect the axis of the gear. This permits the shafts to be extended so that they may pass each other if required. The pitch surface of a hypoid gear is theoretically a hyperboloid of revolution (such a surface is generated by revolving a straight line about another straight line to which it is neither parallel nor intersecting) but actually it is more conical in nature. This type of gear was developed by the Gleason Works, Rochester, New York. It finds application in the rear axle drive of automobiles, as well as elsewhere.

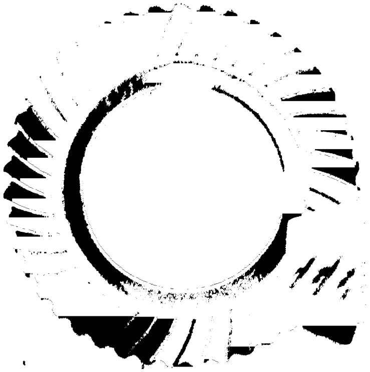


Fig 149 Hypoid Gear and Pinion
Courtesy of Gleason Works, Rochester, New York

QUESTIONS AND PROBLEMS

1. State the law of gearing.
2. When are the profiles of mating teeth said to be conjugate to each other?
3. State the different kinds of curves that are used for gear-tooth profiles.
4. Construct the involute of a circle whose diameter is 3 inches. Carry the construction through one-half the circumference of the circle.
5. Construct a cycloid, using a generating circle whose diameter is 1¼ inches. Let the generating circle roll through a full revolution, giving a completed cycloidal curve.

6. Construct an epicycloid and a hypocycloid. Use a generating circle whose diameter is 1¼ inches and have it roll through a half revolution with a circle 4 inches in diameter. (See Fig. 113.)

7. What kind of a pitch surface has each of the following types of gears: spur, bevel, annular and helical.

8. State three types of gears that are used to connect parallel shafts.

9. State three types of gears that are used to connect shafts whose axes are non-parallel and non-intersecting.

10. State two types of gears that are used for connecting shafts whose axes intersect.

11. Define the following gear terms: *pitch circle, root circle, addendum circle, addendum, backlash, whole depth, face of tooth, flank of tooth* and *face of gear.*

12. A 5-pitch gear has a 4-inch pitch diameter. Find the number of teeth on the gear. *Ans.* 20 teeth.

13. Find the pitch diameter of a 45-tooth gear which has a circular pitch of 1¼ inches. *Ans.* 17.905 in.

14. A 33-tooth cut spur gear has a diametral pitch of 3. Complete the following dimensions; (*a*) circular pitch (*b*) pitch diameter (*c*) addendum circle diameter (*d*) root circle diameter (*e*) total depth of tooth (*f*) backlash (*g*) clearance. *Ans.* (*a*) 1.0472 in. (*b*) 11 in. (*c*) 11⅔ in. (*d*) 10.229 in. (*e*) 0.719 in. (*f*) 0. (*g*) 0.0523 in.

15. Using the tooth proportions for a cast gear, obtain the following dimensions of a spur gear which has 40 teeth and a ¾-inch circular pitch; (*a*) pitch diameter (*b*) addendum (*c*) dedendum (*d*) backlash (*e*) clearance (*f*) thickness of tooth (*g*) width of space. *Ans.* (*a*) 9.549 in. (*b*) 0.24 in. (*c*) 0.2925 in. (*d*) 0.03 in. (*e*) 0.0525 in. (*f*) 0.36 in. (*g*) 0.39 in.

16. State the kinds of curves which are used for the faces and flanks of the teeth of a cycloidal gear.

17. Define the following terms: *pressure angle, angle of obliquity, arc of action, angle of action, arc of approach, angle of approach, arc of recess, angle of recess,* and *pitch point.*

18. (*a*) What is the difference between the path of the point of contact in the cycloidal system and that in the involute system? (*b*) In which of these systems is the pressure angle constant?

19. State the advantages of the involute system of gear-tooth profiles.

20. What is an interchangeable set of gears?

21. Under what conditions are cycloidal spur gears interchangeable?

22. Under what conditions are involute spur gears interchangeable?

23. The smallest gear (or base) of an interchangeable set of cycloidal spur gears is a pinion having 15 teeth. If the circular pitch of the set is ¾ inch, what is the diameter of the generating circle of the entire set. *Ans.* 1.790 in.

24. A 33-tooth cycloidal spur gear has a diametral pitch equal to 3. In drawing the teeth of this gear by Grant's Odontograph method, what radii would be used for faces and flanks? *Ans.* 0.80 in.; 1.27 in.

25. If the teeth of the gear of the preceding problem were involute in form instead of cycloidal, what radii would Grant's Involute Odontograph give for the faces and flanks? *Ans.* 1.42 in.; 1.00 in.

26. When the base circle of an involute gear lies between the base and root circles, how are the flanks completed from the base circle to the root circle?

27. Using Grant's Cycloidal Odontograph, draw the teeth of a 30-tooth, 3-pitch cut spur gear. The teeth need be constructed through only a half circumference of the gear. Use scale full size. (See Fig. 122.)

28. Using Grant's Involute Odontograph, draw the teeth of a 24-tooth, 3-pitch cut spur gear. The teeth need be constructed through only a half circumference of the gear. Use scale full size. (See Fig. 127.)

29. Draw the teeth of a cut cycloidal spur gear which has 18 teeth and a diametral pitch of 2. Use Grant's Method and scale full size.

30. Draw the teeth of a cut involute spur gear which has 18 teeth and a diametral pitch of 2. Use Grant's Method and scale full size.

31. What is meant by interference in involute gears?

32. What is a stub-tooth gear?

33. Two mating spur gears, A and B, have 27 and 45 teeth respectively. What is the angular velocity ratio between A and B? *Ans.* $^5/_3$.

34. Gear A of the preceding problem makes 300 revolutions per minute. Find the r.p.m. of gear B. *Ans.* 180 r.p.m.

35. The pitch radius of gear A of Problem *33* is 9 inches. What is the pitch radius of gear B? *Ans.* 15 in.

36. Does formula (**56**) hold true for all types of gears?

37. Do formulas (**53**) and (**54**) hold true for all types of gears?

38. What is a (*a*) *stepped gear* (*b*) *helical spur gear* (*c*) *twisted gear* (*d*) *herringbone gear* (*e*) *helical gear* (*f*) *worm and worm-wheel* (*g*) *bevel gear* (*h*) *spiral bevel gear* (*k*) *hypoid gear* (*m*) *annular gear* (*n*) *rack gear?*

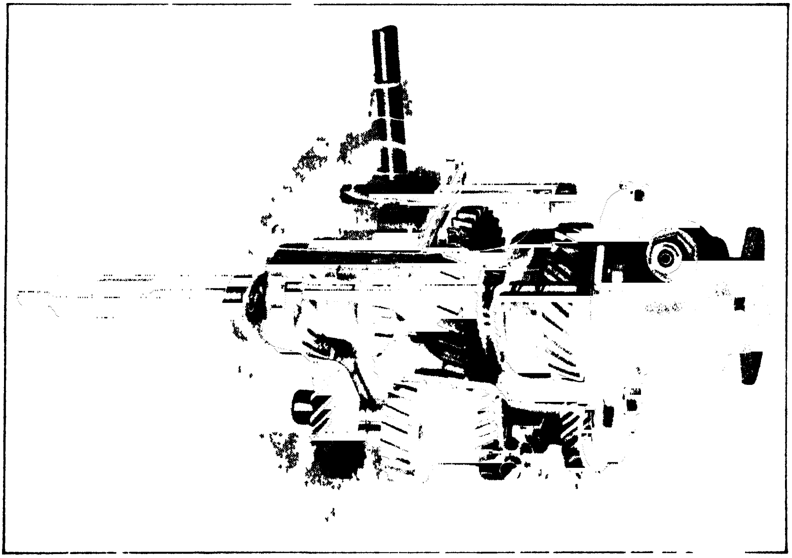
39. State the major characteristics of (*a*) helical spur gear and (*b*) helical gear.

40. A double-thread worm drives a worm-wheel having 120 teeth. When the worm is making 1000 revolutions per minute, what is the rotative speed of the worm-wheel? *Ans.* $16\frac{2}{3}$ r.p.m.

41. Define the following terms used with bevel gears: (*a*) *mounting distance* (*b*) *backing* (*c*) *cone distance* (*d*) *back cone* (*e*) *pitch angle* (*f*) *formative radius* (*g*) *formative number of teeth.*

42. What is Tredgold's Approximation?

43. Following the data as given in the text for the bevel gears of Figs. 146 and 147, reproduce these figures to scale full size.



TRANSMISSION

CHAPTER IX

TRAINS OF MECHANISM

Introduction. In producing the required motion of some element of a machine it often happens that several different mechanisms must be introduced between the source of motion and the driven link. These mechanisms may be of a variety of forms such as gears, cams, pulleys and belts, sprockets and chains, friction wheels, linkages, screws, etc., each of which is introduced to play some definite part in the modification and transmission of the motion. Such a series of mechanisms is known as a *Train of Mechanism.*

A train of mechanism that is made up entirely of gears and other wheels for transmitting continuous rotary motion is called a *Wheel Train.* Such a train is shown in Fig. 150. In this figure, spur gears *A* and *B* are keyed to shafts *1* and *2*, pulleys *C* and *D* are keyed to shafts *2* and *3*, and pulleys *E* and *F* are keyed to shafts *3* and *4* respectively. Thus shaft *2* derives its rotation from shaft *1* through spur gears *A* and *B*, shaft *3* derives its rotation from shaft *2* through pulleys *C* and *D* connected by the crossed belt as shown, and shaft *4* obtains its rotation from shaft *3* through pulleys *E* and *F* which are connected by an open belt. All shafts are supported of course by bearings which are parts of the fixed member or frame. It should be noted that while the shafts of this figure are parallel to each other, other relative positions are often assumed by them. The types of wheels to be used in the train will depend upon such factors as relative positions of shafts, distance between centers, necessity of positive driving, etc.

A *Gear Train* is a wheel train that is composed only of toothed wheels (gears). No wrapping connectors or friction drives are in use in such a train. Hence positive driving is secured throughout its extent. The several different types of gears permit the shafts of such a train to assume various relative positions. (See Figs. 151, 158 and 171.) The gear train of Fig. 151 makes use of three pairs of spur gears to connect the four parallel shafts of the train. The spur gears in use here may be either straight-tooth or helical type in so far as the relative motion is concerned.

Mechanical Efficiency. In order to perform the useful task for which it was designed, a machine requires a supply of energy to be delivered to it from some external source. This source is generally some form of prime mover, such as a steam engine, a gas engine, or an electric motor. On the other hand, the muscular energy of man or the energy of the wind may be utilized. Whatever the source of the energy input (or work input), the machine utilizes part of this energy

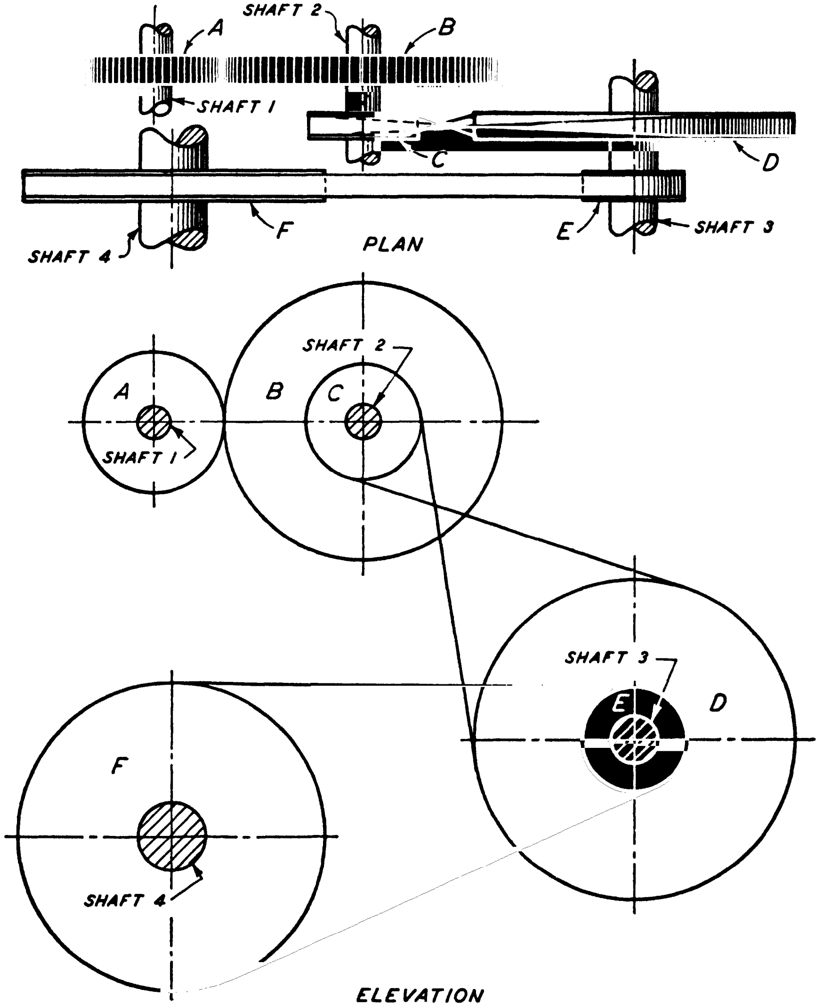


Fig. 150. Wheel Train

in its own operation and the rest, which is by far the larger part, in doing its useful work. The smaller the amount of energy it uses for its own operation, the higher will be its mechanical efficiency, for

$$\text{Mechanical Efficiency} = \frac{\text{output of useful work per unit of time}}{\text{input of work (energy) per unit of time}}$$

In any case, however, the energy input of a machine per unit of time is equal to the sum of its useful work and the work that is done (or the energy consumed) in its own operation. Since power is the rate at which work is done, or the rate at which energy is supplied, the power input of a machine is equal to its total power output. In other words, no machine can gain power.

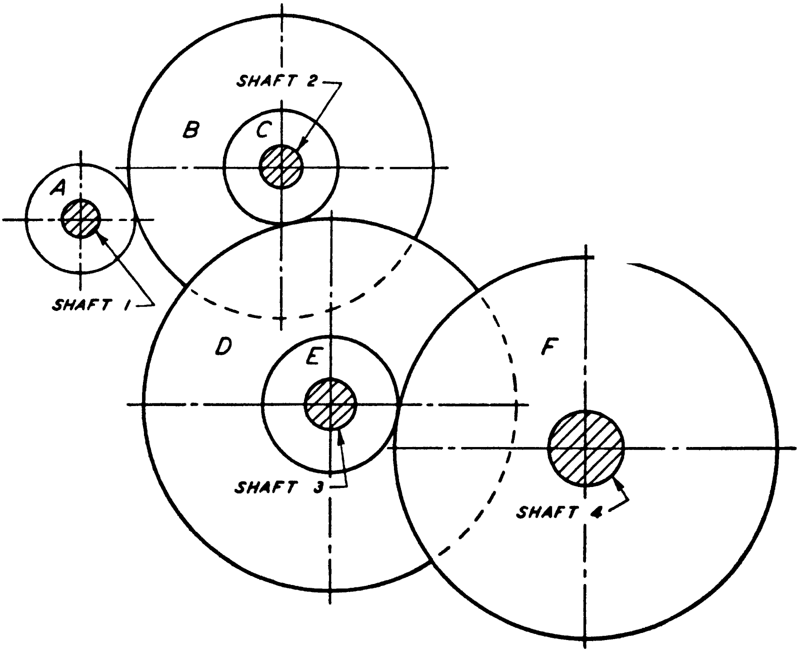


Fig 151 Gear Train, Triple-Reduction

Mechanical Advantage and Speed Reduction. The last statement of the preceding article might be thought to be in contradiction to the fact that a machine can perform a heavy task when supplied with a small amount of power, or that many machines are driven by a very small motor or are even manually operated. For instance, a man using a screw jack can easily raise a large load such as an automobile. This is not because the jack can gain power while it is operated, but rather because the machine is so designed that it possesses a *Mechanical Advantage.*

Work is defined as the product of force and the distance through which it moves, that is:

$$Work = force \times distance$$

Power is defined as work done per unit of time. Hence

$$Power = \frac{work}{time} = \frac{force \times distance}{time}$$

$$= force \times \frac{distance}{time}$$

Therefore from formula (1),

$$Power = force \times velocity \qquad \text{(a)}$$

It is thus seen that power is the product of a force factor and a velocity factor. Neglecting, for the sake of clarity, the energy consumed by the machine in its own operation, or, in other words, assuming that the mechanical efficiency of the machine is 100 per cent, it may be said that power input is equal to the useful power output. Then, from step (*a*), power input is equal to force input multiplied by the velocity of its point of application. Likewise, from step (*a*), power output is equal to force output multiplied by the velocity of its point of application. Since these two products are equal, it is evident that if the velocity factor of the power output is smaller than the velocity factor of the power input, the force factor of the power output must be greater than the force factor of the power input. This condition is produced in machines by including in their design a train of mechanism which reduces the speed and therefore increases the force factor of the power output. The force factor of the power output is equal to the resistance which is overcome by the machine in performing its useful task. The mechanical advantage of a machine is then defined as the ratio of the resistance overcome to the force applied.*

The erroneous statement that a machine can gain power (that is, increase the power delivered to it) is no doubt due to the fact that it possesses a mechanical advantage.

Value of a Gear Train. The value of any wheel train may be defined as the ratio of the r.p.m., or angular velocity of the first wheel

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*See the article on Torque Multiplication in *Machine Design*, by Winston, published by The American Technical Society.

or shaft to the r.p.m., or angular velocity of the last wheel or shaft of the train. Due to the fact that most wheel trains are designed to reduce speed so as to give a mechanical advantage, the value of a train is often spoken of as the speed-reduction factor. In such a case, the train's value is of course greater than unity.

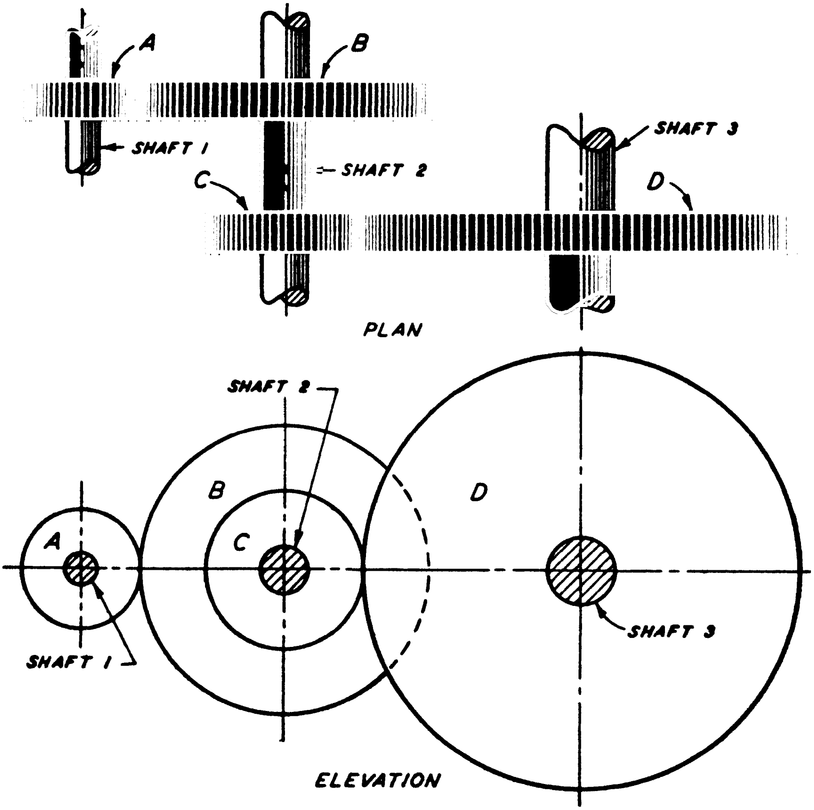


Fig. 152. Double-Reduction Gear Train

The gear train of Fig. 152 will first be considered. In this train, shafts *1* and *2* are connected by gears *A* and *B*, and shafts *2* and *3* are connected by gears *C* and *D*. It should be noted that *B*, which is driven by *A*, and *C*, which is the driver of *D*, are both keyed to shaft *2*. Therefore gears *B* and *C* and shaft *2* all have the same r.p.m. or

$$N_b = N_c = N_2 \tag{a}$$

Applying formula (56) to the pair of gears, *A* and *B*,

$$\frac{N_a}{N_b}=\frac{V_{aa}}{V_{ab}}=\frac{t_b}{t_a} \text{ or } \frac{N_1}{N_2}=\frac{t_b}{t_a} \tag{b}$$

Applying formula (56) to the pair of gears, C and D,

$$\frac{N_c}{N_d}=\frac{V_{ac}}{V_{ad}}=\frac{t_d}{t_c} \text{ or } \frac{N_2}{N_3}=\frac{t_d}{t_c} \tag{c}$$

Equating the product of the first members of steps (*b*) and (*c*) to the product of the second members of the same steps,

$$\frac{N_a}{N_b}\times\frac{N_c}{N_d}=\frac{N_1}{N_2}\times\frac{N_2}{N_3}=\frac{t_b}{t_a}\times\frac{t_d}{t_c} \tag{d}$$

But from step (*a*), $N_b=N_c$, therefore by cancellation

$$\frac{N_a}{N_d}=\frac{N_1}{N_3}=\frac{t_b}{t_a}\times\frac{t_d}{t_c}=T, \text{ the train value} \tag{68}$$

Since B and D are driven gears, and A and C are drivers, formula (68) states that the value of a gear train, $\frac{N_a}{N_d}$ or $\frac{N_1}{N_3}$, is equal to the continued product of the numbers of teeth of the driven gears, divided by the continued product of the numbers of teeth of the driving gears.

If a train contains more than two pairs of gears, the second member of formula (68) will be extended to include, as factors, a ratio for each pair. It should be noted carefully that, in every ratio of the second member of formula (68), the numbers of teeth of the driven gears are in the numerator while those of the driving gears are in the denominator. Since formula (56) holds true for all types of gears, formula (68) likewise is applicable to all gear trains.

With many types of gears, the pitch radii or pitch diameters are directly proportional to the numbers of teeth. When such is the case, such a factor as $\frac{t_b}{t_a}$ can be replaced by $\frac{R_{pb}}{R_{pa}}$ or $\frac{D_{pb}}{D_{pa}}$.

From formula (68),

$$N_a=TN_d$$

or

$$N_d=\frac{N_a}{T} \tag{69}$$

Formula (69) states that in a gear train the r.p.m. of the last driven shaft is equal to the r.p.m. of the first driving shaft divided by the value of the train. This statement is applicable, not only to all gear trains, but also to all wheel trains.

Example 1. Obtain the value of a double-reduction spur gear train like that of Fig. 152 when

$$t_a = 15 \text{ teeth}$$
$$t_b = 60 \text{ teeth}$$
$$R_{pc} = 8 \text{ inches}$$
$$R_{pd} = 28 \text{ inches}$$

Solution. With a pair of spur gears, the radii of the pitch circles are directly proportional to the numbers of teeth. Therefore

$$\frac{t_d}{t_c} = \frac{R_{pd}}{R_{pc}}$$

Substituting in formula (68)

$$T = \frac{t_b}{t_a} \times \frac{R_{pd}}{R_{pc}}$$

Evaluating $$T = \frac{60}{15} \times \frac{28}{8} = \frac{4}{1} \times \frac{7}{2} = 14 \quad \textit{Ans.}$$

Example 2. If in the preceding example the r.p.m. of shaft *1* is 300, what is the r.p.m. of shaft *3*?

Solution. Applying formula (69)

$$N_3 = N_d = \frac{N_a}{T} = \frac{300}{14} = 21.43 \text{ r.p.m.} \quad \textit{Ans.}$$

Example 3. Obtain the value of the train of Fig. 151 and the r.p.m. of the last driven shaft when the following information is given:

$N_1 = 750$ r.p.m.
$t_a = 16$ teeth
$t_b = 64$ teeth
$t_c = 20$ teeth
$t_d = 90$ teeth
$D_{pe} = 5$ inches
$D_{pf} = 25$ inches

Solution. Applying formula (68)

$$T = \frac{t_b}{t_a} \times \frac{t_d}{t_c} \times \frac{t_f}{t_e}$$

Since *E* and *F* are spur gears (either straight-toothed or helical)

$$\frac{D_{pf}}{D_{pe}} = \frac{t_f}{t_e}$$

Therefore $$T = \frac{t_b}{t_a} \times \frac{t_d}{t_c} \times \frac{D_{pf}}{D_{pe}}$$

Evaluating $$T = \frac{64}{16} \times \frac{90}{20} \times \frac{25}{5} = 90 \quad \textit{Ans.}$$

From formula (69)

$$N_4=\frac{N_1}{T}=\frac{750}{90}=8\tfrac{1}{3}\text{ r.p.m.}\quad \textit{Ans.}$$

Value of a Wheel Train. Referring to the wheel train of Fig. 150, it will be noted that gear A and pulleys C and E are drivers, while gear B and pulleys D and F are driven links, or followers. Assume that the numbers of teeth, t_a and t_b, of gears A and B, and the diameters D_c, D_d, D_e, and D_f, of pulleys C, D, E, and F, respectively, are known.

From Formula (56)

$$\frac{N_a}{N_b}=\frac{N_1}{N_2}=\frac{t_b}{t_a} \tag{a}$$

From formula (26)

$$\frac{N_c}{N_d}=\frac{N_2}{N_3}=\frac{D_d}{D_c} \tag{b}$$

Also from formula (26)

$$\frac{N_e}{N_f}=\frac{N_3}{N_4}=\frac{D_f}{D_e} \tag{c}$$

From steps (*a*), (*b*) and (*c*)

$$\frac{N_1}{N_2}\times\frac{N_2}{N_3}\times\frac{N_3}{N_4}=\frac{N_a}{N_b}\times\frac{N_c}{N_d}\times\frac{N_e}{N_f}=\frac{t_b}{t_a}\times\frac{D_d}{D_c}\times\frac{D_f}{D_e} \tag{d}$$

From step (*d*) by cancellation,

$$\frac{N_1}{N_2}\times\frac{N_2}{N_3}\times\frac{N_3}{N_4}=\frac{N_1}{N_4}$$

and

$$\frac{N_a}{N_b}\times\frac{N_c}{N_d}\times\frac{N_e}{N_f}=\frac{N_a}{N_f}$$

Hence step (*d*) becomes

$$\frac{N_1}{N_4}=\frac{N_a}{N_f}=\frac{t_b}{t_a}\times\frac{D_d}{D_c}\times\frac{D_f}{D_e} \tag{70}$$

Therefore

$$T=\frac{t_b}{t_a}\times\frac{D_d}{D_c}\times\frac{D_f}{D_e} \tag{71}$$

The second member of formula (71) will contain as many factors (ratios) as there are pairs (a driver with its follower) of wheels. It should be noted that, as in formula (68), the numerators of these factors involve data concerning the driven links, while the denominators involve data relative to the drivers. This is due to the fact that these individual reduction factors are in each case inversely proportional to the angular velocities, as shown by steps (*a*), (*b*) and (*c*).

Idler Gear. A comparison of the spur-gear trains of Figs. 152 and 153 shows that while they both employ three shafts, one more gear is used in the train of Fig. 152 than in that of Fig. 153. This is due to the fact that two gears are attached to shaft *2* of the former, while only one gear is attached to the second shaft of the latter. By this arrangement of gears in Fig. 153 gear *A* drives gear *B* on the intermediate shaft. Gear *B*, in turn, effects the rotary motion of shaft *3* by driving gear *C*. Such a gear as *B*, which acts simultaneously as both a driving gear and a driven gear, is called an *Idler*.

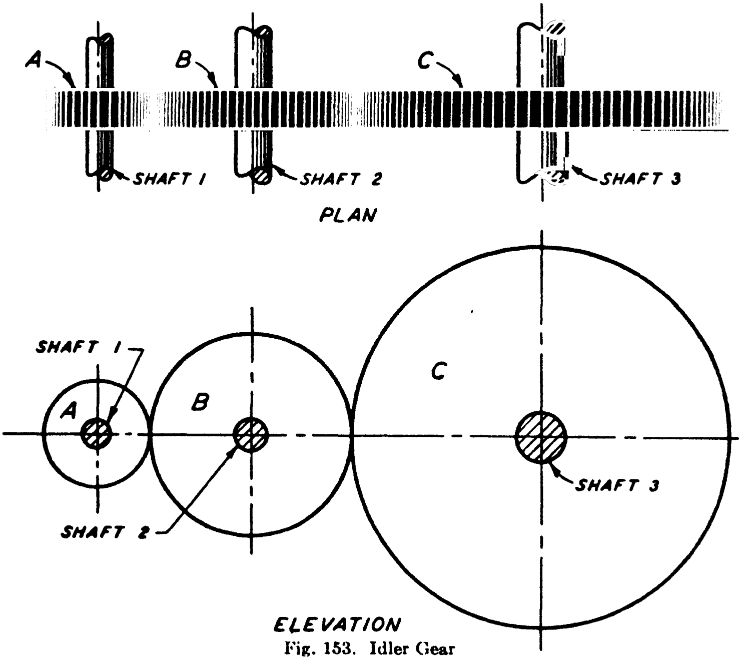


Fig. 153. Idler Gear

Referring again to Fig. 153, investigation is made of the affect of an idler upon the angular velocity (r.p.m. ratio) of the gears between which it is placed. Since gear *B* is driven by gear *A*, from formula (**56**)

$$\frac{N_a}{N_b}=\frac{t_b}{t_a} \qquad \text{(a)}$$

In a similar manner for gears *B* and *C*, which are paired with each other,

$$\frac{N_b}{N_c}=\frac{t_c}{t_b} \qquad \text{(b)}$$

From steps (*a*) and (*b*)

$$\frac{N_a}{N_b}\times\frac{N_b}{N_c}=\frac{t_b}{t_a}\times\frac{t_c}{t_b}$$

Simplifying
$$\frac{N_a}{N_c}=\frac{t_c}{t_a} \qquad (c)$$

If the idler were omitted from the train of gears and *A* placed directly in mesh with *C*, the angular velocity ratio between them would be the same as that given by step (*c*). Therefore when an idler is used in a train of gears, it does not affect the value of the train. However if *A*, rotating clockwise, were directly in mesh with *C*, the latter would rotate in a counterclockwise direction. It is evident from Fig. 153 that, with *B* placed between *A* and *C*, and with *A* still rotating in a clockwise direction, idler *B* will rotate oppositely and hence cause *C* to rotate in a clockwise direction. It is therefore evident that when an idler is placed between two gears it reverses the direction of rotation of the driven gear, causing it to rotate in the same direction as the driver with no change in the angular velocity ratio.

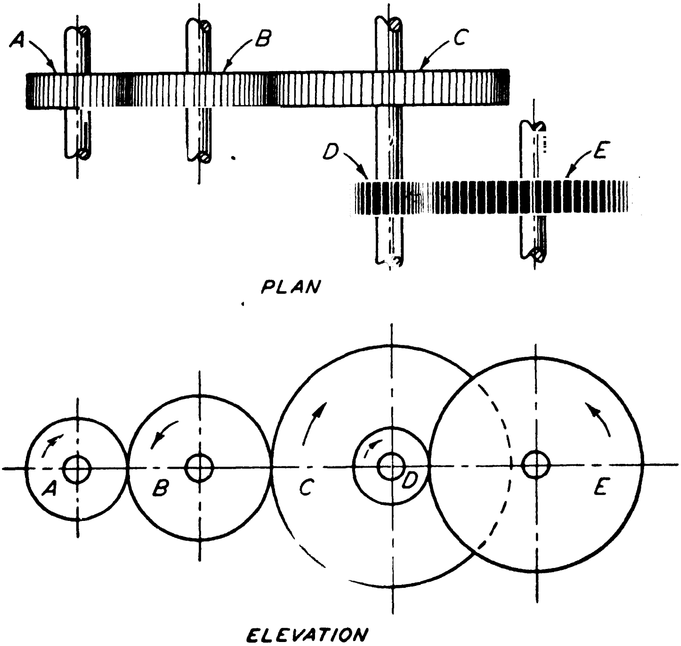


Fig. 154. Gear Train with Idler

The use of an idler gear makes it possible to employ smaller spur gears in connecting a pair of parallel shafts with a fixed and unalterable distance between centers. This statement can be verified by assuming that two shafts such as *1* and *3* of Fig. 153 are 15 inches between centers and that the ratio $\frac{N_1}{N_3}$ is equal to 4. Consider first that the two gears, *A* and *C*, are used without an idler. To secure their diameters,

$$R_{pa}+R_{pc}=15 \qquad \text{(a)}$$

$$\frac{R_{pa}}{R_{pc}}=\frac{N_c}{N_a}=\frac{1}{4}$$

or

$$R_{pc}=4R_{pa} \qquad \text{(b)}$$

Substituting $4R_{pa}$ for R_{pc} in step (*a*)

$$R_{pa}+4R_{pa}=15$$

$$5R_{pa}=15$$

$$R_{pa}=3 \text{ in.}$$

From step (*b*)

$$R_{pc}=4\times 3=12 \text{ in.}$$

Therefore the two diameters are

$$D_{pa}=6 \text{ in.}$$

and

$$D_{pc}=24 \text{ in.}$$

Consider next that gears *A* and *C* are used with an idler, and that gear *A* in this case is given a diameter of 3 inches. Since the idler does not influence the angular velocity ratio, it is evident from formula (54) that

$$D_{pc}=D_{pa}\times\frac{N_a}{N_c}=D_{pa}\times\frac{N_1}{N_3}$$

$$=3\times 4=12 \text{ inches}$$

Obviously, from Fig. 153, the pitch diameter of the idler,

$$D_{pb}=15-(R_{pa}+R_{pc})$$

$$=15-(1\tfrac{1}{2}+6)=7\tfrac{1}{2} \text{ in.}$$

Hence with a 7½-inch idler, gears *A* and *C* have pitch diameters of 3 inches and 12 inches, while without this idler they require the larger pitch diameters of 6 inches and 24 inches respectively.

Example. A gear train containing an idler, *B*, is shown in Fig. 154. Obtain the value of the train and the r.p.m. of the last driven gear when gear *A* makes 90 revolutions per minute. The gears have pitch diameters and numbers of teeth as follows:

A, 6 -inch diameter, 24 teeth
B, 9 -inch diameter, 36 teeth
C, 15 -inch diameter, 60 teeth
D, 4½-inch diameter, 18 teeth
E, 13½-inch diameter, 54 teeth

Solution. In this train, B (the idler), C, and E are the driven gears, and therefore the given data applying to them will be introduced into the numerator of the train value. A, B (the idler), and D are driving gears and their given data will be introduced into the denominator. Using the numbers of teeth in the formula for the train value,

$$T=\frac{t_b}{t_a}\times\frac{t_c}{t_b}\times\frac{t_e}{t_d}$$

$$=\frac{36}{24}\times\frac{60}{36}\times\frac{54}{18}=7.5 \quad \textit{Ans.}$$

Or, using the pitch diameters to obtain the train value,

$$T=\frac{D_{pb}}{D_{pa}}\times\frac{D_{pc}}{D_{pb}}\times\frac{D_{pe}}{D_{pd}}$$

$$=\frac{9}{6}\times\frac{15}{9}\times\frac{13.5}{4.5}=7.5 \quad \textit{Ans.}$$

It should be noted that in both of the above equations for T, the data for the idler occurs in both numerator and denominator. Therefore, as previously shown, the idler could have been neglected in obtaining the value of the train.

Applying formula (69)

$$N_e=\frac{N_a}{T}$$

and evaluating

$$N_e=\frac{90}{7.5}=12 \text{ r.p.m.} \quad \textit{Ans.}$$

Directional Relation of a Gear Train. Due to the variety of ways in which various gears influence the direction of rotation of their followers, no set rule can be made for obtaining the directional relation between the first driving gear and the last driven gear of all gear trains. Therefore the best procedure is to establish the directional relation by considering in turn the direction of rotation given by each driver to its immediate follower in gear trains as well as all trains of mechanism.

In order to demonstrate this, let gear *A* of Fig. 154 be given a clockwise direction of rotation as shown. Gear *B* will then be caused by *A* to rotate in a counterclockwise direction. *B*, in turn, will give to *C* a clockwise rotation. Since gear *D* is keyed to the same shaft as *C*, it will be similarly directed and so cause the last driven gear *E* to rotate in a counterclockwise direction.

Tumbler Gears. A simple arrangement of gears shown in Fig. 155 (*a*) and (*b*) is one of several types of gear trains by which the driven gear can be given either direction of rotation. In this train, *A* is the driver and *D* the driven, each being keyed to its shaft. Two idler gears *B* and *C*, called *Tumbler Gears* in this train, are carried on an arm *F*, which is pivoted on the shaft of *D*. *C* is always in mesh with both *B* and *D*. The arm *F*, when moved about its pivot to the position shown in Fig. 155 (*a*), brings *B* in mesh with *A*, so the driving is from *A* through both *B* and *C* to *D*. This position of the arm and its tumblers gives to *D* a direction of motion opposite to that of *A*, as shown in the figure by the arrows. When the arm is turned to the position shown in Fig. 155 (*b*), *C* is brought into mesh with the driver

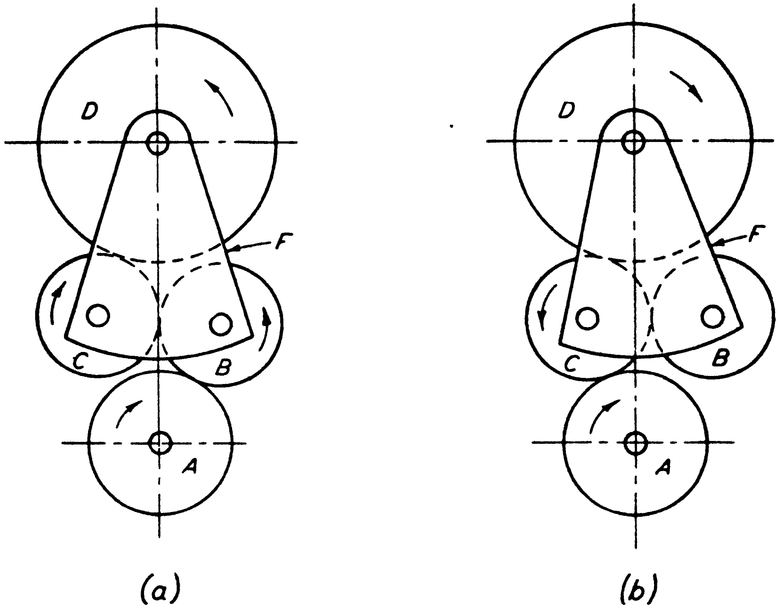


Fig 155 Tumbler Gears

A, and B is taken out of mesh with A, so that the driving is from A through C to D. Since in this position only one idler is involved, D is given the same direction of rotation as A. Because the tumblers are idler gears, the angular velocity ratio of A to D is the same as if A were directly in mesh with D.

Reverted Gear Train. The arrangement of gears shown in Fig. 156 is called a *Reverted Gear Train.* In such a train the axes of driving and driven shafts, *1* and *3*, are coincident. These shafts are connected by a train of gears through an intermediate shaft, *2*. The axis of the latter is parallel to the coincident axes of shafts *1* and *3*. Fastened to the intermediate shaft are gears B and C. B is driven by A, which is keyed to the driving shaft, and C drives D, which is keyed to the driven shaft of the train. In this manner, the train is provided with two speed reductions. Therefore its total reduction ratio (or train value) is the product of its two reductions, or from formula (68)

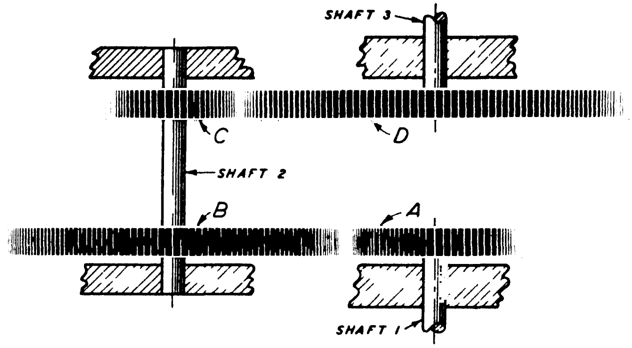


Fig 156 Reverted Gear Train

$$T=\frac{N_1}{N_3}=\frac{N_a}{N_d}=\frac{t_b}{t_a}\times\frac{t_d}{t_c}=\frac{R_{pb}}{R_{pa}}\times\frac{R_{pd}}{R_{pc}}$$

It is evident from Fig. 156 that the sum of the pitch radii R_{pa} and R_{pb} must equal the sum of the pitch radii R_{pc} and R_{pd}, this sum being in turn equal to the distance between the parallel axes of the intermediate shaft and the driving and driven shafts.

Back Gears. It is necessary in the case of many machine tools, as well as other machines, that the shaft or spindle of the machine be driven at several different speeds. Hence some speed change mechanism must be installed between the shaft of the machine and the source of motion. Where the machine is driven from a constant-speed countershaft (usually suspended from the ceiling above the machine), cone pulleys are commonly used to afford such speed changes. It is evident, as shown heretofore in this text, that a pair of cone pulleys will afford as many different speeds to the driven shaft as there are pairs of steps on the pulleys. In order to provide a still greater range of speeds, a reverted gear train is often employed in conjunction with the driven stepped cone pulley. A reverted gear train when so installed is spoken of as a set of *Back Gears*. By their installation, twice as many speeds can be obtained as would be afforded by the cone pulleys alone.

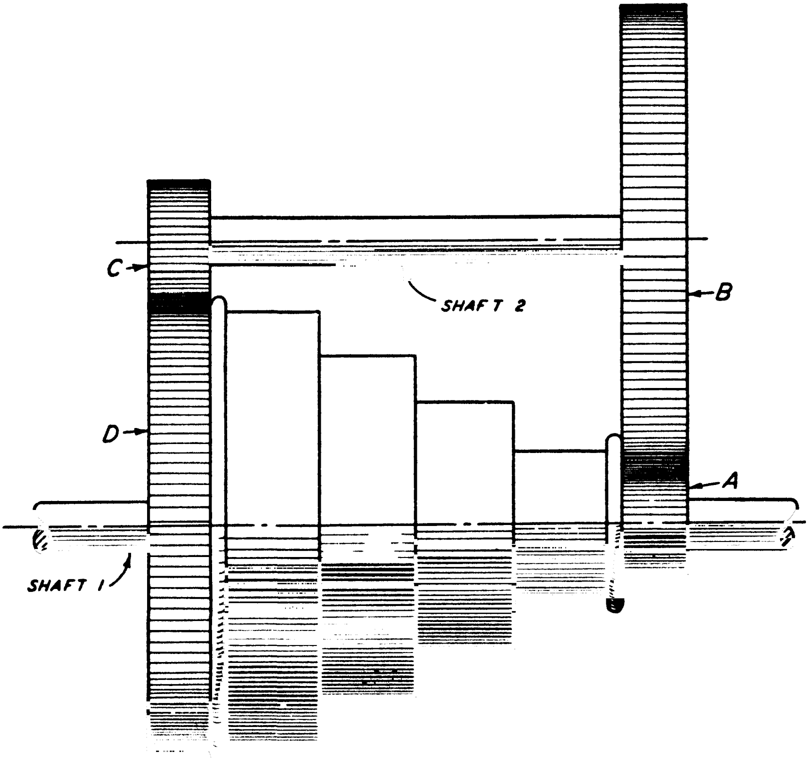


Fig 157 Cone Pulley with Back Gears

In Fig. 157 is shown a typical installation of a train of back gears in use with a cone pulley located on the spindle of an ordinary lathe. The back gears consist of pinion *A*, which is attached to and hence rotates with the cone pulley; gear B and pinion C, both of which are attached to shaft *2*; and gear D which is keyed to the spindle *1*. Pinion A and the cone pulley rotate freely on the spindle. Gear D may be fastened to, or unfastened from, the cone pulley. The back gears may be engaged or disengaged by a lever. They are shown engaged (or in mesh) in the figure.

When the back gears are disengaged, gear D is fastened to the cone pulley. This in turn connects the latter with the spindle so that any speed of the pulley is communicated directly and unchanged to the spindle. Since the pulley has four steps, it has four speeds to give to the spindle.

When the back gears are engaged, A and B, as well as C and D, are in mesh as shown in the figure. Gear D is now unfastened from the cone pulley so that the latter, with its attached pinion A, drives gear D and hence spindle *1* through B and C. In this manner the spindle may receive four more different speeds which are in each case one of the four pulley speeds affected or changed by the value of the back gear train.

It is thus seen that while the cone pulley may receive only one speed for each position of the belt, the spindle may receive two speeds, one with the back gears disengaged or out of service and one with the back gears engaged. Thus the number of possible speeds of the spindle is double that of the cone pulley if back gears are provided. Therefore in the case of Fig. 157 the spindle is provided with eight speeds. Since the train of back gears furnishes a double reduction, each of the four spindle speeds obtained while the back gears are engaged is lower than that speed received with the same belt position but with the back gears disengaged. In fact for each belt position, the spindle speed with back gears engaged is equal to the spindle (or pulley speed) with back gears disengaged, divided by the train value T which is $\frac{N_a}{N_d}$. See formulas (68) and (69). These spindle speeds are generally arranged in a geometric progression, as is shown by the following example.

Example. The spindle of a lathe is to be furnished with eight speeds arranged in a geometric progression. The minimum spindle speed is 10 r.p.m. and the maximum, 500 r.p.m. The speed change mechanism is to consist of stepped cone pulleys and back gears. It is required to obtain the following:

(*a*) the number of steps on each cone pulley
(*b*) the ratio of the geometric progression of the speeds
(*c*) the numerical values of the speeds
(*d*) the value of the back gear train

Solution. (*a*) Since back gears are to be used, the number of steps on each cone pulley will be equal to one-half the number of spindle speeds. Hence

$$8 \div 2 = 4 \text{ steps} \quad \textit{Ans.}$$

(*b*) The ratio of a geometric progression is given by the formula

$$k = \left(\frac{N_x}{N_1}\right)^{\frac{1}{x-1}} \qquad (72)$$

in which

k = the ratio of the geometric progression
N_x = the maximum speed
N_1 = the minimum speed
x = the number of speeds

Here $N_x = N_8 = 500$ r.p.m., $N_1 = 10$ r.p.m., $x = 8$. Evaluating in the formula for k,

$$k = \left(\frac{500}{10}\right)^{\frac{1}{8-1}} = 50^{\frac{1}{7}} = 1.749 \quad \textit{Ans.}$$

The value of $50^{\frac{1}{7}}$ or $\sqrt[7]{50}$ is obtained in the following manner:

$$\text{Log } 50 = 1.69897$$
$$\text{Log } \sqrt[7]{50} = \frac{1.69897}{7} = 0.24271$$
$$\therefore \quad \sqrt[7]{50} = \text{Log}^{-1}\ 0.24271 = 1.749$$

(*c*) The speeds are as follows:

$N_1 = 10$ r.p.m.
$N_2 = kN_1 = 1.749 \times 10 = 17.49$ r.p.m.
$N_3 = kN_2 = 1.749 \times 17.49 = 30.58$ r.p.m.
$N_4 = kN_3 = 1.749 \times 30.58 = 53.47$ r.p.m.
$N_5 = kN_4 = 1.749 \times 53.47 = 93.51$ r.p.m.
$N_6 = kN_5 = 1.749 \times 93.51 = 163.51$ r.p.m.
$N_7 = kN_6 = 1.749 \times 163.51 = 285.93$ r.p.m.
$N_8 = kN_7 = 1.749 \times 285.93 = 500.00$ r.p.m.

(*d*) Since the above speeds are in a geometric progression, the following continued proportion holds true:

$$\frac{N_8}{N_4}=\frac{N_7}{N_3}=\frac{N_6}{N_2}=\frac{N_5}{N_1}$$

or

$$\frac{500}{53.47}=\frac{285.93}{30.58}=\frac{163.51}{17.49}=\frac{93.51}{10}=9.35$$

Since there is this constant ratio existing between the pairs of speeds as shown, it is evident that the stepped cone pulleys can be designed to give the four higher speeds, N_8, N_7, N_6, and N_5. These pulley speeds will become the spindle speeds with the back gears disengaged. Then the back gear train may be designed to furnish a train value of 9.35; so that, with the back gears engaged, the cone pulley speeds will be reduced (in order) to the spindle speeds, N_4, N_3, N_2, and N_1.

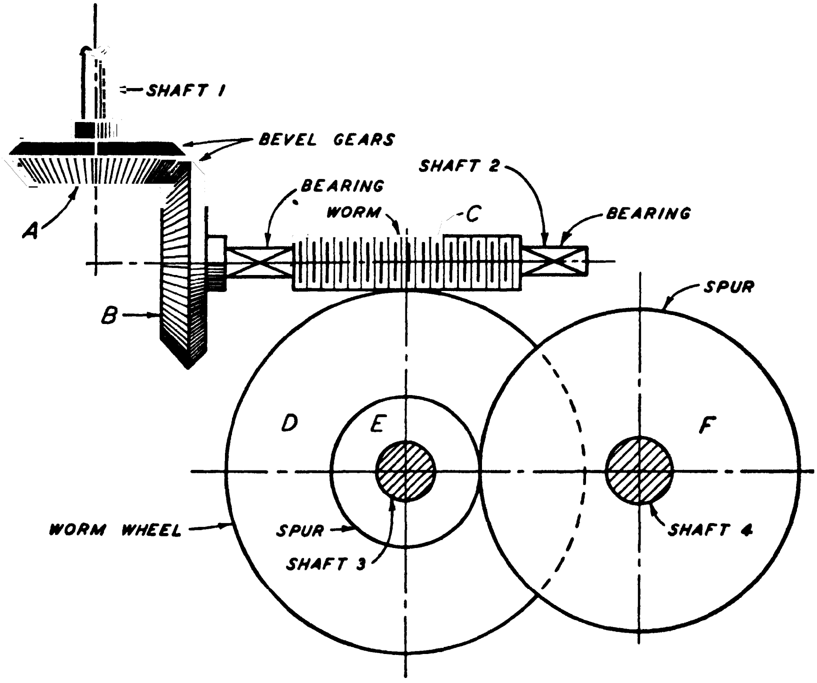


Fig. 158. Gear Train with Nonparallel Shafts

With $T=9.35$, referring to Fig. 157, from formula (68)

$$T=\frac{N_b}{N_a}\times\frac{N_d}{N_c}=9.35$$

$\frac{N_b}{N_a}$ and $\frac{N_d}{N_c}$ are the r.p.m. ratios of the two pairs of gears of the train. These ratios may be given any values whose product is 9.35. If each is given the same value so as to equalize the reductions in each pair,

$$\frac{N_b}{N_a}=\frac{N_d}{N_c}=\sqrt{9.35}=3.06$$

Gear Trains Connecting Non-Parallel Shafts. It has been previously stated that trains of gears (or mechanisms) can be used to connect other than parallel shafts. In so doing, gears other than straight-toothed and helical spur gears must of course be used. Such a train is shown in Fig. 158. Here the first driving shaft, *1* of the train, drives shaft *2* through the bevel gears A and B. Shaft *2* drives shaft *3* by means of the worm C and the worm wheel D. Shaft *4*, the last driven shaft of the train, secures its motion through the pinion E and the gear F. The relative location of the shafts is clearly shown in the figure.

In order to demonstrate that the value of this train can be obtained in the same manner as the value of a train which is used to connect parallel shafts, the following example is introduced dealing with Fig. 158.

Example. Let it be assumed that:

$N_1=N_a=675$ r.p.m.

$t_a=24$ teeth

$t_b=36$ teeth

Worm, C, carries a double thread.

$t_d=60$ teeth

D_p of E, or $D_{pe}=6$ in.

D_p of F, or $D_{pf}=18$ in.

Solution. Applying formula (68) to this train,

$$T=\frac{t_b}{t_a}\times\frac{t_d}{t_c}\times\frac{t_f}{t_e} \tag{a}$$

Since with spur gears,

$$\frac{t_f}{t_e}=\frac{D_{pf}}{D_{pe}}$$

step (*a*) becomes

$$T=\frac{t_b}{t_a}\times\frac{t_d}{t_c}\times\frac{D_{pf}}{D_{pe}} \tag{b}$$

Evaluating in step (*b*) (a double-threaded worm is a gear of 2 teeth, hence $t_c=2$ teeth)

$$T=\frac{36}{24}\times\frac{60}{2}\times\frac{18}{6}=135, \text{ the train value.} \quad \textit{Ans.}$$

From formula (69)

$$N_4=N_f=\frac{N_1}{T}=\frac{675}{135}=5 \text{ r.p.m.} \quad \textit{Ans.}$$

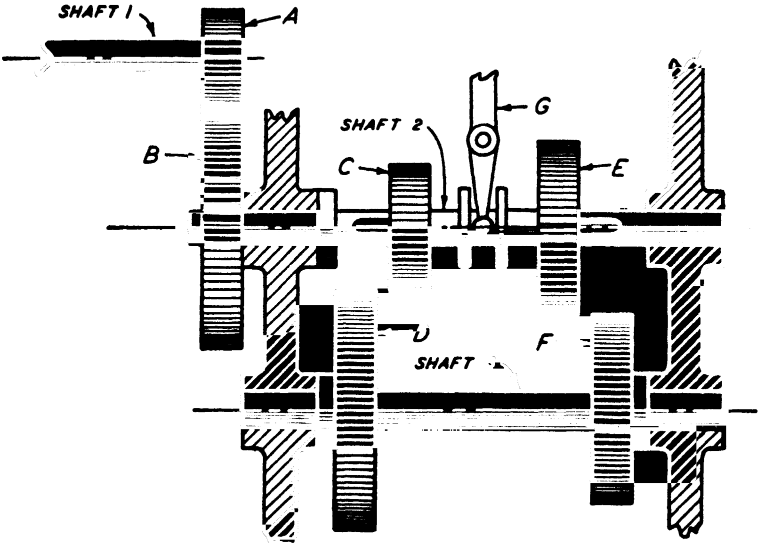


Fig. 159. Sliding Gears

Sliding Gears. When it is necessary to provide two different speeds to a driven shaft, a speed-change mechanism employing *Sliding Gears* may be used. In Fig. 159, which illustrates this type of a mechanism, *C* and *E* are the sliding gears. They are constructed as a single unit and are fastened to shaft *2* by a feather key. The latter is a key which permits the gears to be moved axially along their shaft, but, like other keys, prevents them from turning on the shaft. The sliding or shifting of the gears is effected by the shifting lever *G*. By moving *C* and *E* to the left, *C* is thrown into mesh with *D*, permitting shaft *3* to be driven by shaft *2* through gears *C* and *D*. By moving *C* and *E* to the right along their shaft, *E* is brought into mesh with *F*, so that now shaft *3* is driven by shaft *2* through the gears *E* and *F*. Hence when *C* is in mesh with *D*, the constant-speed drive shaft, *1*, drives shaft *2* through *A* and *B*, and then shaft *3* from *2* through *C* and *D*. With *C* out of mesh with *D* and with *E* in mesh with *F*, shaft *1* still drives *2* through *A* and *B*, but *3* is now driven from *2* through *E* and *F*.

An application of this device is to be found in the transmissions of automobiles as well as in the gear trains of many other machines.

Speed Reducers. In many cases efficient plant operation has for some time dictated the adoption of the individual machine drive in place of the group drive. The group drive, as its name implies, makes use generally of one prime mover from which the power is derived for the machines of the entire plant or a large part thereof. It is evident then that, in such a drive, use is necessarily made of long lines of shafting connected to each other, and to the machines driven by them, principally by numerous belts and pulleys. The individual machine drive of course uses a prime mover, generally an electric motor, to drive a single machine. The advent of the high-speed electric motor gave great impetus to the development of the individual machine drive. With its high rotative speed reduced by a train of mechanism to the lower speed of the machine, a mechanical advantage is obtained which permits of the use of a lower horsepower. In order that this installation may occupy as little space as possible and at the same time give the desired reduction in speed, the train of mechanism employed is generally a train of gears housed compactly in a gear box or case. This type of gear train is known to the trade as a *Speed Reducer*.

Speed reducers use practically all kinds of gears. Some use spur gears which may be straight-toothed, single or double helical, or annular. Others use bevel or worm gears, while still others use different types of gears within the same unit.

The principal advantages of the use of speed reducers, as compared with other arrangements of trains of mechanisms, are as follows:

1. Conservation of space by allowing the installation of productive equipment formerly occupied by the more extensive trains of mechanism

2. Ease and completeness of lubrication, either by the continuous splash system or by a self-contained system of forced lubrication for high-speed units

3. Gear cases eliminate trouble from oil leaks and drippings and keep the units free from dust, grit, etc.

4. Complete covering of moving parts eliminates working hazards and thus prevents injury to plant personnel.

5. Positive driving and constant reduction ratios provided.

6. Uninterrupted service and freedom from repairs and adjustments provided, thus preventing expensive production delays.

Types of Speed Reducers. A few representative types of speed reducers will be introduced and some information will be given on each type. The student should examine the illustrations carefully to note details of construction.

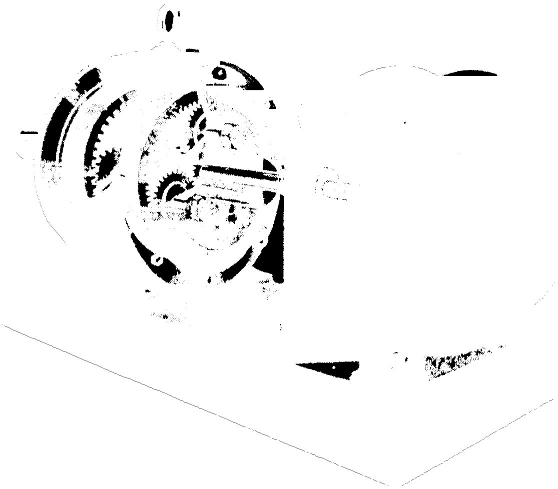


Fig 160 Spur Gear Reducer (Double Reduction)
Courtesy of Foote Bros Gear and Machine Corp, Chicago Ill

A spur gear reducer is illustrated in Fig. 160. The high-speed shaft of this reducer is coupled directly to the shaft of the motor and carries a pinion which meshes with three idler gears. These idlers rotate on fixed studs which are fastened either to one end plate or (in the case of a second set of idlers for double-reduction units) to a partition cast integral with the frame. The idlers in turn drive an annular wheel. In single-reduction units, this annular wheel is fastened to the slow-speed or driven shaft of the unit. In the double-reduction type shown in Fig. 160, the shaft to which this annular is fastened is an intermediate shaft that carries a pinion which drives another annular wheel through a second group of idlers, and the final annular wheel is fastened to the low-speed shaft of the reducer. It will be noted that the axes of the first driving shaft (high-speed shaft) and last driven shaft (low-speed shaft) are coincident. Since idler gears have no effect on the train value, it is evident that each of the two reductions thus provided is equal to the number of teeth on the annular wheel, divided by the number of teeth on the pinion. Therefore the total reduction, or train value, of the double-reduction reducer is equal to the product of its two individual reductions. The type shown in Fig. 160 is built to furnish reductions from 7 to 1 up to 10 to 1. Other types provide total reductions as high as 100 to 1 and up.

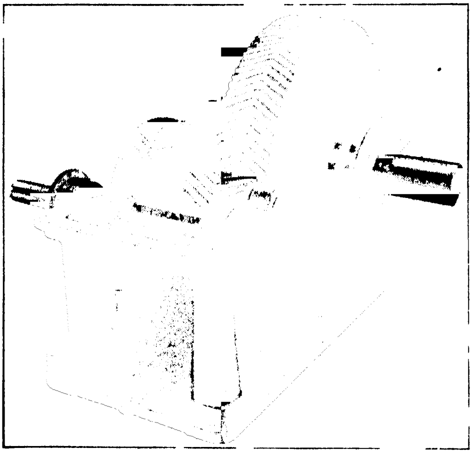


Fig 161 Herringbone Gear Reducer
(Double Reduction)
Courtesy of Link-Belt Company, Chicago, Ill

A speed reducer which uses herringbone gears is illustrated in Fig. 161. As in several of the following illustrations, the cover of the gear case is removed in order to disclose the mechanism. The first driving or high-speed shaft of the train, carrying a pinion, is shown extending through the gear case to the left; while the last driven shaft (the low-speed shaft) carrying the largest gear, extends through its bearing in the case to the right. Between these two parallel shafts is the intermediate shaft to which is fastened both a pinion and a gear. In this manner two individual gear reductions are afforded, so that the reducer is a double-reduction type. It can be obtained to give reductions from 10 to 1 up to 70 to 1. Single and triple-reduction reducers of the herringbone type are also available.

Fig. 162 shows two views of a double-reduction worm-gear reducer. An inspection of this figure clearly discloses the relative location of the shafts and the two pairs of worm gears which make possible the double reduction. Since a single pair of worm gears can be designed to yield a rather high reduction in itself, it is evident that two pairs in the train of a speed reducer can give to the reducer a tremendous speed-reduction or train value. The speed reducer of Fig. 162 can be obtained to give reduction ratios from 25 to 1 to 10,000 to 1. It should be remembered that even a 10,000 to 1 reduction requires that each pair of a double-reduction gear train furnish a reduction of only 100 to 1, assuming that the two pairs are designed to give equal reductions. In the right-hand view of Fig. 162, the high-speed shaft carrying a worm gear is seen extending through the gear case to the right while the low-speed shaft extends through the case to the left. The left-hand view of this figure exposes the intermediate shaft in full and shows thereon the worm and worm wheel which are fastened to it.

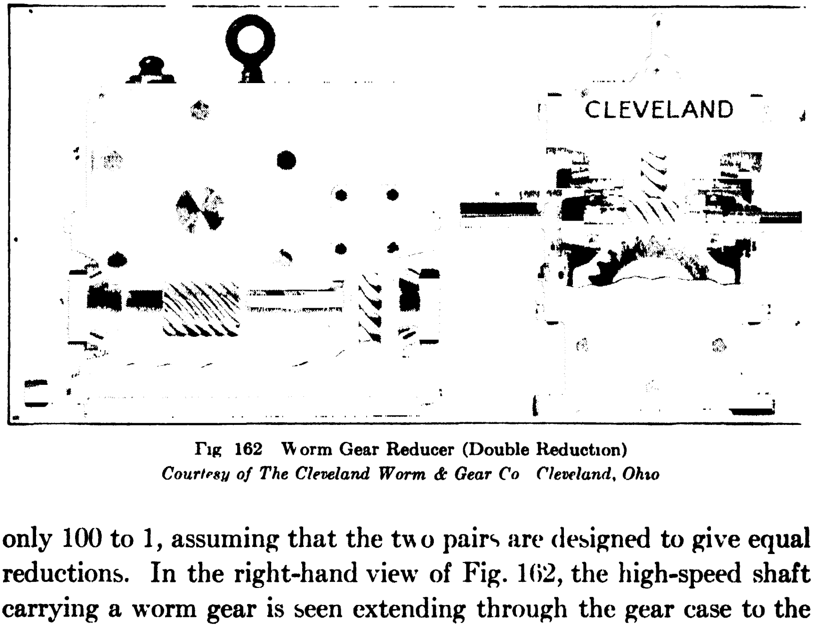


Fig 162 Worm Gear Reducer (Double Reduction)
Courtesy of The Cleveland Worm & Gear Co Cleveland, Ohio

In the illustrations of speed reducers which have been presented so far, a single type of gears was used in each. In some reducers more than one type is used, as is shown in Figs. 163 and 164. Both of these double-reduction reducers obtain one reduction by using a pair of spiral bevel gears and the other reduction by using a pair of helical spur gears. The reducer of Fig. 163 is said to be a vertical right-angle type because the low-speed (driven) shaft is vertically placed and the angle between the low-speed and high-speed (driving) shaft is a right angle. Fig. 164 is called a horizontal right-angle reducer because its

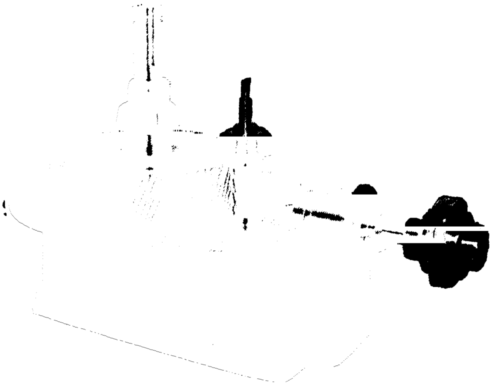


Fig 163 Vertical Right-Angle Reducer
Courtesy of The Falk Corporation, Milwaukee, Wis

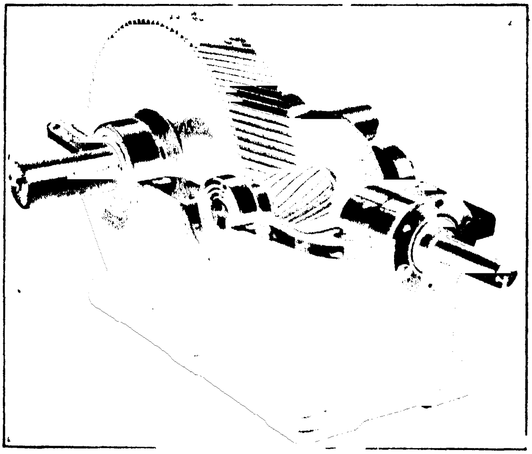


Fig 164 Horizontal Right-Angle Reducer
Courtesy of The Falk Corporation, Milwaukee, Wis

low-speed shaft is in a horizontal position, and the angle between its low-speed and high-speed shafts is a right angle.

Epicyclic Gear Trains. Fig. 165 (*a*) will be recognized as the simplest kind of gear train, containing only one pair of gears. Here the meshing gears B and C rotate about their respective axes O and

O_1 with respect to A, the fixed or stationary link which supports them.

The same kinematic chain shown in Fig. 165 (*a*) is given in Fig. 165 (*b*). In the latter, however, gear B, as indicated by the cross-hatching, is the fixed link instead of arm A. Gear C, while still rotating about O_1, rotates with arm A about O, the center of B. This substitution of B as the fixed link in the place of A has changed the kinematic chain from the ordinary gear train of Fig. 165 (*a*) to another type of mechanism known as an *Epicyclic Gear Train.* Then it may be stated that an epicyclic gear train is one in which one gear is fixed and the other gears of the train, carried by a revolving arm, rotate not only about their own centers but also about the center of the fixed gear. Due to this double rotation of its gears, this train is also known as a *Planetary Gear Train.* It is easily identified by its fixed gear and rotating arm. However it presents itself in a multitude of forms, due to the number of gears and the several types thereof that may be introduced into its design.

Angular Velocity Ratios in Epicyclic Trains. In epicyclic gear trains it is required to obtain, in particular, the ratio of the angular velocities of the last gear of the train and of the arm by which it is carried. Of course each angular velocity in this ratio is with respect to the fixed link or frame of the mechanism. In determining them it will be necessary to distinguish between clockwise rotation and counterclockwise rotation. This will be accomplished by using the positive (+) sign for clockwise and the negative (−) sign for counterclockwise. Thus if it happens that the two angular velocities involved in the ratio are of the same sign, the ratio is positive. This itself then indicates that the two links are rotating in the same direction. On the other hand, if the angular velocities have unlike signs their ratio is negative; hence such a ratio indicates opposite directional relation.

The simple train of Fig. 165 (*b*) (reproduced in Fig. 166) is next investigated. As stated previously, gear B is fixed (or stationary) and arm A rotates about O, the axis of B. Arm A carries gear C, which therefore must rotate about O. Since gear C is in mesh with gear B, it is caused also to rotate about its own axis, O_1. It is the combination of these two rotations that produces its angular velocity with respect to the frame of the epicyclic train.

A knowledge of the value of the angular velocity ratio between gear C and arm A may be obtained in three steps by a method entirely general, which may be followed in the solution of all epicyclic gear trains. Refer to Fig. 166.

Step 1. Release gear *B* from the frame so that it is no longer fixed. Lock gears *B* and *C* so that they cannot rotate relative to arm

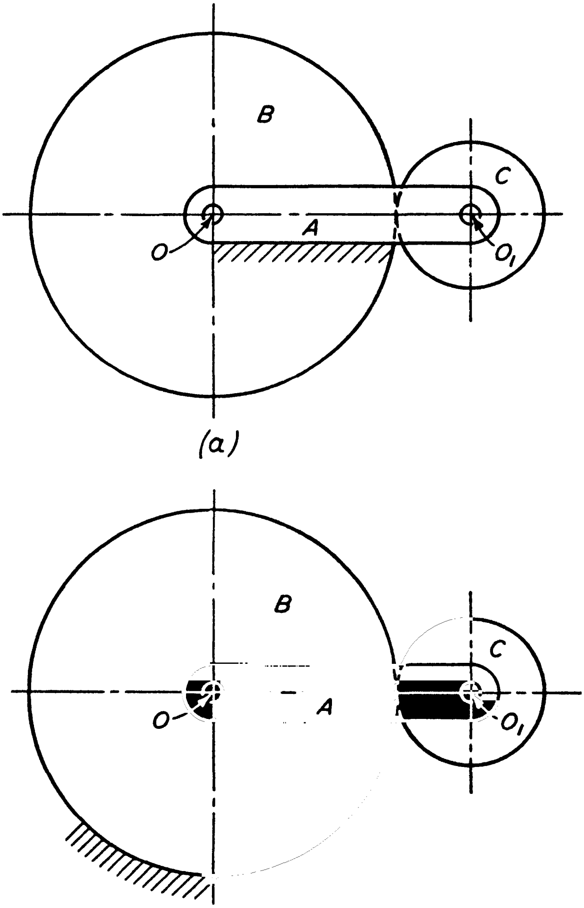


Fig. 165

A. Rotate arm *A* with the gears through one revolution about the fixed axis *O*. During this turn, the locked mechanism takes the several dotted positions shown in the figure. Follow point M_1 which is located on the pitch circle of gear *C*. After ¼ turn it is at M_2. After the succeeding ¼ turns it takes the positions M_3, M_4, and M_1. The point thus returns to its initial position, having completed, with its gear *C*, one revolution in a clockwise direction. It is evident then that *C* made one clockwise revolution about the center O_1 Since gear *B* was also locked with the train during this rotation, it likewise has made one clockwise revolution with the arm. Let the turns be represented by *N* as in the preceding work of this text, even though the time unit is not admitted. So when

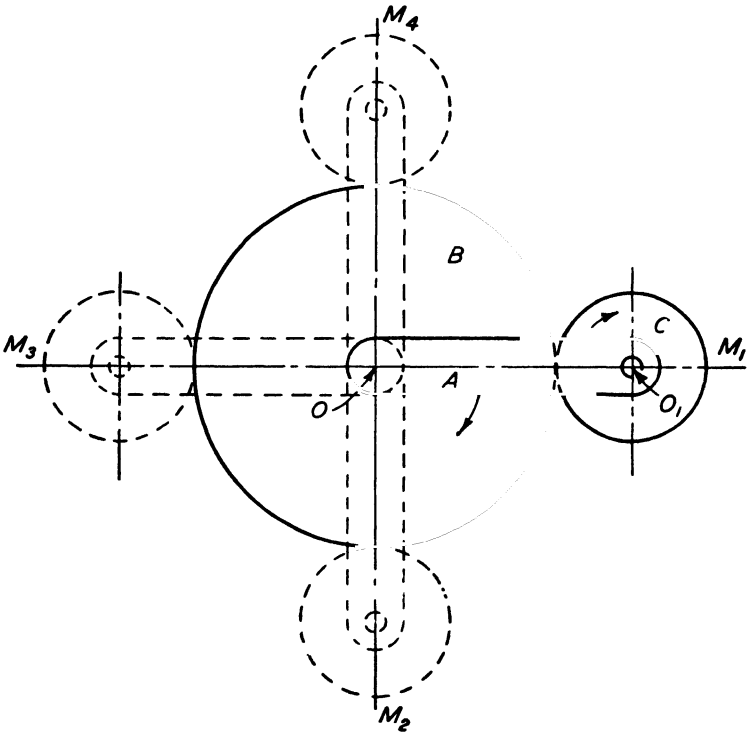


Fig. 166

$$N_a = +1$$

it follows that

$$N_c = +1 \text{ and } N_b = +1$$

Step 2. Unlock gears *B* and *C* with gear *B* still released from the frame, and let *A* be taken as the fixed member of the train. The latter will now act as an ordinary gear train. Let t_b and t_c represent, as usual, the numbers of teeth on *B* and *C* respectively. Now rotate gear B about its axis O through one counterclockwise revolution (while A remains stationary) *in order to return it to its initial position.* (In step *1* it had one clockwise turn.) As B makes this counterclockwise revolution, C (which is in mesh with B) will make $\frac{t_b}{t_c}$ revolutions in a clockwise direction. Hence in this step with

$$N_b = -1$$

it follows that

$$N_c = +\frac{t_b}{t_c} \text{ and } N_a = 0$$

Note. $N_a = 0$ because arm A is functioning as the fixed link in this step.

Step 3. The total or resultant number of turns of each link with respect to the fixed axis O is then the sum of the turns obtained in steps *1* and *2*. Performing this addition for each link of the epicyclic train gives

$$N_a = +1 + 0 = +1$$
$$N_b = +1 + (-1) = 1 - 1 = 0$$
$$N_c = +1 + \left(+\frac{t_b}{t_c}\right) = 1 + \frac{t_b}{t_c}$$

These results, representing the total number of turns of each link, show that gear B is the stationary member, with $N_b = 0$; arm A is making one clockwise revolution, for $N_a = +1$; and gear C is making a number of clockwise revolutions equal to $1 + \frac{t_b}{t_c}$.

Therefore the angular velocity ratio of gear C to arm A is given by:

$$\frac{N_c}{N_a} = \frac{1 + \frac{t_b}{t_c}}{1}$$

Since in this ratio the values of N_c and N_a are both positive, the sign of their ratio is positive. This also indicates that both A and C rotate in the same direction.

Example. Let it be assumed that gears B and C of Fig. 166 have 80 teeth and 20 teeth respectively. Find the angular velocity ratio between gear C and arm A.

Solution. Here $t_b = 80$ teeth and $t_c = 20$ teeth. Since for this simple epicyclic train it has just been proved that in general

$$\frac{N_c}{N_a}=\frac{1+\frac{t_b}{t_c}}{1}$$

evaluating,

$$\frac{N_c}{N_a}=\frac{1+\frac{80}{20}}{1}=\frac{1+4}{1}=\frac{5}{1}\quad Ans.$$

Therefore gear C will make 5 revolutions to each revolution of the arm A, while turning in the same direction.

Tabulation of Results. The immediate recording of the results of each of the three steps involved in the solution of an epicyclic gear train aids materially in effecting that solution. The table shown was prepared to accompany the foregoing example.

Step No.	Number of Revolutions		
	N_a	N_b	N_c
1	$+1$	$+1$	$+1$
2	0	-1	$+\frac{t_b}{t_c}=+\frac{80}{20}$
3, Totals	$+1$	0	$1+\frac{t_b}{t_c}=1+(+\frac{80}{20})=+5$

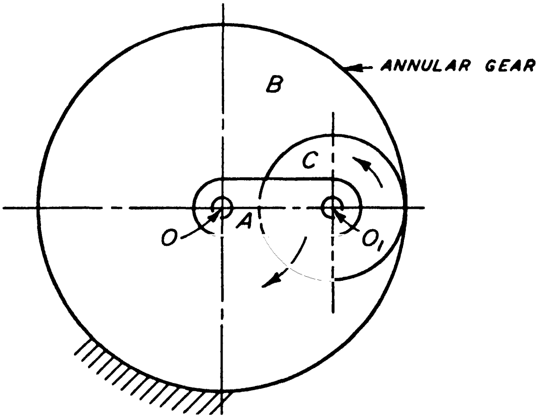


Fig 167

Simple Epicyclic Train with Annular. As previously stated, positive rotation of the arm A of the simple epicyclic train of Fig. 166

produces positive rotation of gear C. If the fixed gear B is replaced by an internal or annular gear as shown in Fig. 167, positive rotation of the arm A will cause gear C to have a negative rotation as shown by the arrows. But with gears B and C having like numbers of teeth in both of these mechanisms, there will be not only a different directional relation but also a different angular velocity ratio between C and A. To demonstrate this, the steps of the general solution will be applied to the mechanism of Fig. 167, with t_b and t_c equal to 80 teeth and 20 teeth, respectively, as before. The tabulated solution follows.

Step No.	Number of Revolutions		
	N_a	N_b	N_c
1	+1	+1	+1
2	0	−1	$-\frac{t_b}{t_c} = -\frac{80}{20} = -4$
3, Totals	+1	0	$+1+(-4) = -3$

The totals show that gear C makes three counterclockwise turns to each clockwise turn of arm A. The angular velocity ratio of C to A is:

$$\frac{N_c}{N_a} = \frac{-3}{+1} = -3$$

Simple Epicyclic Train with Idler. In Fig. 168 is shown an idler, D, placed in mesh with gears B and C of Fig. 166. This idler produces the same changes in directional relation and angular velocity ratio that are produced by substitution of the annular in Fig. 167. This is proved by the following tabulated solution in which t_b and t_c are the same numbers as before (for comparative purposes) and t_d is equal to any number of teeth, say 16.

Step No.	Number of Revolutions			
	N_a	N_b	N_c	N_d
1	+1	+1	+1	+1
2	0	−1	$-\frac{t_b}{t_c} = -\frac{80}{20} = -4$	$+\frac{t_b}{t_d} + \frac{80}{16} = +5$
3, Totals	+1	0	$+1+(-4) = -3$	$+1+(+5) = 1+5 = +6$

It will be noted from these totals that the resulting angular velocity ratio is

$$\frac{N_c}{N_a}=\frac{-3}{+1}=-3$$

which is the same as that obtained in the case of the use of an annular gear.

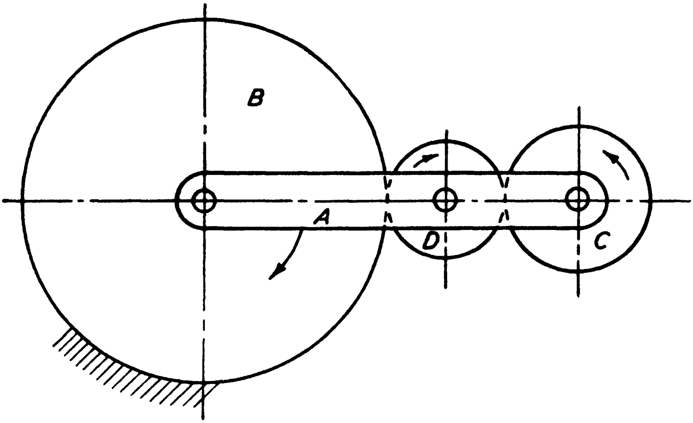


Fig. 168

Other Epicyclic Gear Trains. A few more examples of epicyclic trains will be introduced in order to show that the method of solution for simple gear trains can be just as easily applied to those which are more complicated.

Example 1. In the train of Fig. 169 gear B is fixed, and arm A rotates clockwise and carries the idlers C and D. An annular gear, E, is in mesh with D. The axes of rotation of A and E are coincident with the axis of the fixed link B. Obtain the angular velocity ratio of E to A when

$$t_b = 36 \text{ teeth}$$
$$t_c = 18 \text{ teeth}$$
$$t_d = 24 \text{ teeth}$$
$$t_e = 120 \text{ teeth}$$

Solution. Step 1. Lock the gears, release gear B from the frame, and give arm A one clockwise revolution. Then with

$$N_a = +1$$

it follows that

$$N_b = +1,\ N_c = +1,\ N_d = +1,\ N_e = +1$$

Step 2. Unlock the gears and, with arm A fixed, give to gear B one counterclockwise revolution. Then with

$$N_b = -1 \text{ and } N_a = 0 \ (A \text{ is fixed in this step})$$

it follows that

$$N_c = +\frac{t_b}{t_c} = +\frac{36}{18} = +2$$

$$N_d = -\frac{t_b}{t_d} = -\frac{36}{24} = -1\tfrac{1}{2}$$

Note. Since C is an idler between B and D, like all idlers it may be ignored except for direction.

$$N_e = -\frac{t_b}{t_e} = -\frac{36}{120} = -\tfrac{3}{10}$$

Step 3. The results are now tabulated and the additions made as shown in the row of totals.

Step No.	Number of Revolutions				
	N_a	N_b	N_c	N_d	N_e
1	+1	+1	+1	+1	+1
2	0	−1	$+\frac{t_b}{t_c} = +\frac{36}{18} = +2$	$-\frac{t_b}{t_d} = -\frac{36}{24} = -1\frac{1}{2}$	$-\frac{t_b}{t_e} = -\frac{36}{120} = -\frac{3}{10}$
3, Totals	+1	0	$+1 + (+2) = 1 + 2 = +3$	$+1 + (-1\frac{1}{2}) = +1 - 1\frac{1}{2} = -\frac{1}{2}$	$+1 + (-\frac{3}{10}) = +1 - \frac{3}{10} = +\frac{7}{10}$

From these totals the angular velocity ratio of E to A is

$$\frac{N_e}{N_a} = \frac{+\frac{7}{10}}{+1} = +\frac{7}{10}$$

which shows that the annular gear E rotates in the same direction as the arm A and makes $^7/_{10}$ revolution to 1 revolution of A.

It should be noted that in step *2* it is always the *fixed gear* of the epicyclic train which is given one turn opposite in direction to that given the arm in step *1*. For this reason in the table the total for the fixed gear will always be zero, as it evidently should be.

Example 2. This example will also concern the train of Fig. 169; however, in this case the annular gear E will be taken as the fixed link, and the angular velocity ratio of gear B to arm A will be obtained. The gears are to have the same numbers of teeth as in the preceding example.

Solution. Step 1. Lock the gears, release the fixed link E from the frame, and give to arm A one clockwise turn. With

$$N_a = +1$$

it follows that

$$N_b = +1,\ N_c = +1,\ N_d = +1,\ N_e = +1$$

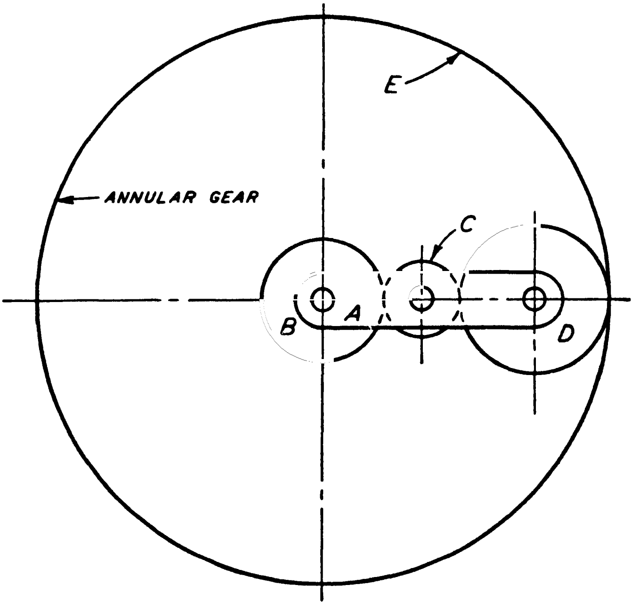


Fig. 169

Step 2. Unlock the gears and, with arm A fixed, give to gear E one counterclockwise turn. Then with

$$N_e = -1 \text{ and } N_a = 0$$

it follows that

$$N_d = -\frac{120}{24} = -5$$

$$N_c = +\frac{120}{18} = +\frac{20}{3} = +6\tfrac{2}{3}$$

$$N_b = -\frac{120}{36} = -3\tfrac{1}{3}.$$

Step No.	Number of Revolutions				
	N_a	N_b	N_c	N_d	N_e
1	+1	+1	+1	+1	+1
2	0	$-\frac{t_e}{t_b}=-\frac{120}{36}$ $=-3\frac{1}{3}$	$+\frac{t_e}{t_c}=+\frac{120}{18}$ $=+6\frac{2}{3}$	$-\frac{t_e}{t_d}=-\frac{120}{24}$ $=-5$	−1
3, Totals	+1	$+1+(-3\frac{1}{3})=$ $+1-3\frac{1}{3}=-2\frac{1}{3}$	$+1+(+6\frac{2}{3})=$ $+1+6\frac{2}{3}=+7\frac{2}{3}$	$+1+(-5)=$ $+1-5=-4$	0

From these totals,

$$\frac{N_b}{N_a}=\frac{-2\frac{1}{3}}{+1}=-2\frac{1}{3}$$

Therefore gear B turns oppositely to arm A and makes $2\frac{1}{3}$ revolutions for each revolution of A.

Example 3. What is the r.p.m. of gear B in the train of Fig. 169 when the annular gear is stationary and arm A is given 90 revolutions per minute?

Solution. Here $N_a=90$ r.p.m.

From the preceding example,

$$\frac{N_b}{N_a}=-2\frac{1}{3}$$

Since the negative sign has been used in interpreting directional relation, it need not be considered further. Hence evaluating

$$\frac{N_b}{90}=2\frac{1}{3}$$

$$N_b=2\frac{1}{3}\times 90=210 \text{ r.p.m.} \quad \textit{Ans.}$$

Reverted Epicyclic Gear Train. In Fig. 170 an epicyclic train is used to connect two shafts whose axes are coincident. Hence, as in the case of ordinary gear trains, this epicyclic train is said to be reverted. In the mechanism of Fig. 170, the arm A is keyed to the driving shaft *1* and therefore becomes the driving link of the train. It carries gears C and D which are keyed to shaft *3* and hence have identical speeds. Gear B, in mesh with C, is fastened to the frame and thus becomes the stationary gear. Gear E, in mesh with D, is keyed to the driven shaft *2* and is therefore the last driven link of the mechanism. This type of epicyclic gear may be designed to give a considerable speed reduction.

In order to become familiar with this reverted train, the angular velocity ratio of gear E (the last driven gear) to arm A (the first driver of the train) will be found. It is assumed that the arm turns in a clockwise direction and that the numbers of teeth of the gears are as follows: $t_b=80$, $t_c=81$, $t_d=80$, and $t_e=81$. The solution is presented in the following tabulation.

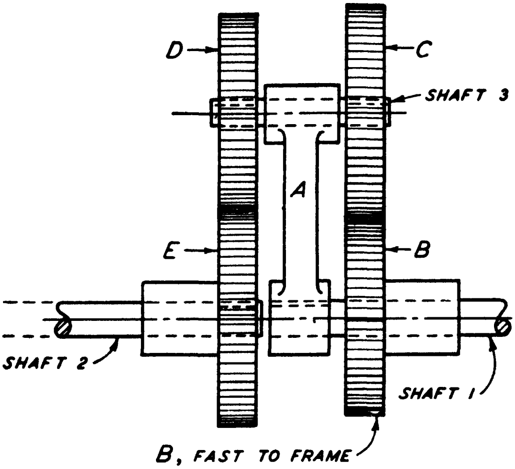


Fig 170 Reverted Epicyclic Train

Step No.	Number of Revolutions			
	N_a	N_b	N_c and N_d	N_e
1	$+1$	$+1$	$+1$	$+1$
2	0	-1	$+\frac{t_b}{t_c}=+\frac{80}{81}$	$-\left(\frac{t_b}{t_c}\times\frac{t_d}{t_e}\right)=-(\frac{80}{81}\times\frac{80}{81})$ $=-\frac{6400}{6561}$
3, Totals	$+1$	0	$+1+(+\frac{80}{81})=+1+\frac{80}{81}$ $=+1\frac{80}{81}$	$+1+(-\frac{6400}{6561})=+1-\frac{6400}{6561}$ $=+\frac{161}{6561}$

From these totals,

$$\frac{N_e}{N_a}=\frac{+\frac{161}{6561}}{+1}=+\frac{161}{6561}$$

which indicates that gear E will turn in the same direction as arm A and will make only $\frac{161}{6561}$ revolution to 1 revolution of the arm; for instance, if the arm A (and shaft *1*) makes 200 revolutions per minute, the r.p.m. of the gear E (and shaft *2*) is, from the above result:

$$N_e = \frac{161}{6561} N_a$$

or

$$N_e = \frac{161}{6561} \times 200 = 4.9 \text{ r.p.m.} \quad \textit{Ans.}$$

QUESTIONS AND PROBLEMS

1. Of what may a train of mechanism be composed?

2. What is a wheel train?

3. Are all wheel trains made up only of toothed wheels?

4. What type of wrapping-connector mechanism may be used in a train of mechanism which must give positive driving?

5. How does a machine utilize the energy with which it is supplied?

6. Define *mechanical efficiency.*

7. How does work differ from power?

8. If a machine is to have a mechanical advantage, must the train of mechanism which it employs be a speed increaser or a speed reducer?

9. Can a machine increase the power delivered to it?

10. Can a machine utilize all the power delivered to it in performing its useful task?

11. What is a prime mover?

12. Define *train value.*

13. The last driven gear of a gear train has a lower r.p.m. than its first driving gear. Is the value of the train more or less than unity?

14. In the double-reduction gear train of Fig. 152, gears A, B, C, and D have 20, 50, 18, and 72 teeth respectively. Shaft *1* rotates at 325 r.p.m. Find

(*a*) the value of the gear train
(*b*) the r.p.m. of shaft *3*
(*c*) the r.p.m. of shaft *2*

Ans. (*a*) 10 (*b*) 32.5 r.p.m. (*c*) 130 r.p.m.

15. If gears A and B of the preceding problem are 4-pitch gears and C and D are 3-pitch gears, find

(*a*) the pitch diameter of each gear of the train
(*b*) the distance between centers of shafts *1* and *2*, and *2* and *3*
(*c*) the linear velocity in f.p.m. at the pitch circle of each gear
(*d*) the circular pitches of each pair of gears

Ans. (*a*) 5 in.; 12.5 in.; 6 in.; 24 in. (*b*) 8¾ in.; 15 in. (*c*) 425.4 f.p.m.; 425.4 f.p.m.; 204.2 f.p.m.; 204.2 f.p.m. (*d*) 0.7854 in.; 1.0472 in.

16. Obtain the train value and the r.p.m. of gear F of the triple-reduction train of Fig. 151 when the r.p.m. of shaft *1* is 750 and $t_a = 16$, $t_b = 48$, $t_c = 20$, $t_d = 70$, $t_e = 24$, and $t_f = 100$ teeth. *Ans.* 43¾; 17.1 r.p.m.

17. In problems *14* and *16* the first driving shafts rotate in a clockwise direction. What is the direction of rotation of the driven shaft of each of these trains?

18. The following list of data is given for the wheel train of Fig. 150:

Gear *A*, $P_d = 3$ $D_p = 6$ in. $N_a = 300$ r.p.m.
Gear *B*, $P_d = 3$ $D_p = 28$ in.
Pulley *C*, $D = 9$ in.
Pulley *D*, $D = 27$ in.
Pulley *E*, $D = 9$ in.
Pulley *F*, $D = 33$ in.

Find the value of the train, and the r.p.m. and the direction of rotation of shaft *4*, with shaft *1* rotating clockwise. *Ans.* 51⅓; 5.8 r.p.m.; clockwise.

19. What is an idler gear?

20. Is the value of a gear train affected by the inclusion of an idler?

21. Does an idler influence the directional relation of a train?

22. In the train of Problem *18* an idler is placed between gears *A* and *B*; otherwise the train is unaltered. Find the r.p.m. and the direction of rotation of the last driven shaft. *Ans.* 5.8 r.p.m.

23. *A*, *B*, and *C* of the gear train of Fig. 153 are 4-pitch gears having pitch diameters of 4 inches, 7 inches, and 20 inches respectively. Find the train value, and the r.p.m. and direction of rotation of gear *C*, with gear *A* rotating at 750 r.p.m. in a counterclockwise direction. *Ans.* 5; 150 r.p.m.; counterclockwise.

24. What are tumbler gears?

25. When is a gear train reverted?

26. A drill press is driven from an overhead constant-speed shaft by a pair of 4-stepped cone pulleys used in conjunction with a set of back gears. The shaft or spindle of the drill press has a minimum speed of 92 r.p.m. and a maximum speed of 920 r.p.m., and its eight speeds are in a geometric progression. Obtain:

(*a*) the ratio of the geometric progression
(*b*) the eight speeds of the shaft of the drill press
(*c*) the train value of the set of back gears
(*d*) the speeds obtained with the back gears disengaged
(*e*) the speeds obtained with the back gears engaged.

Ans. (*a*) 1.39 (*b*) $N_1 = 92$; $N_2 = 128$; $N_3 = 178$; $N_4 = 247$; $N_5 = 343$; $N_6 = 477$; $N_7 = 662$; $N_8 = 920$ r.p.m. (*c*) 3.72 (*d*) The four higher speeds, N_5, N_6, N_7, N_8 (*e*) The four lower speeds, N_1, N_2, N_3, N_4.

Note. Problem *26* at the end of Chapter VI should be reviewed at this time. It is directly related to the preceding problem.

27. In the gear train of Fig. 158 it is given that $t_a = 24$ teeth, $t_b = 40$ teeth, *C* is a triple-threaded worm, $t_d = 81$ teeth, and the pitch diameters of gears *E* and *F* are 6 inches and 24 inches respectively.

(*a*) What is the value of this train of gears?
(*b*) What is the speed of shaft *4*, when shaft *1* is rotating at 720 r.p.m.?

Ans. (*a*) 180 (*b*) 4 r.p.m.

28. The sliding gears *C* and *E*, of Fig. 159, have 16 teeth and 24 teeth, and gears *A*, *B*, *D*, and *F* have 20 teeth, 60 teeth, 48 teeth, and 40 teeth respectively.

(*a*) Find the train value when gear *C* is in mesh with gear *D*.
(*b*) Find the train value when gear *E* is in mesh with gear *F*.

Ans. (*a*) 9 (*b*) 5

29. What is meant by the expression *group drive*?

30. What are the advantages of the so-called speed reducers?

31. If a double-reduction speed reducer employs two individual reductions of 4 to 1 and 6 to 1, what is its total reduction (or train value)? *Ans.* 24 to 1

32. In a double-reduction worm-gear reducer, reductions of 40 to 1 and 50 to 1 are employed. What is the total reduction? *Ans.* 2000 to 1

33. In the layout of Fig. 171, a motor-driven train of gears is used to cause a drum to raise a load. The drum is 24 inches in diameter and $t_a = 1$ (single-threaded), $t_b = 60$, $t_c = 18$, and $t_d = 90$ teeth. The motor speed is 1150 r.p.m. Find

(*a*) the value of the train

(*b*) the r.p.m. of the drum

(*c*) the speed in f.p.m. at which the load is lifted

Ans. (*a*) 300 (*b*) 3.83 r.p.m. (*c*) 24.1 f.p.m.

34. What is an epicyclic train of gears?

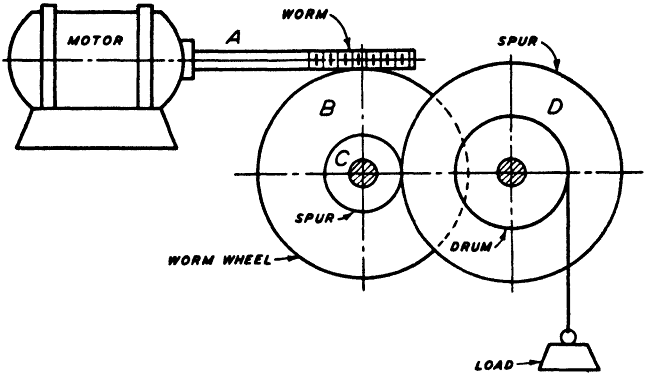


Fig. 171

35. Make a complete tabulation of the number of revolutions of each link and find the angular velocity ratio between the gear C and the arm A of the epicyclic train of Fig. 166. Let the fixed gear B and gear C have 90 teeth and 15 teeth respectively. Assume that arm A turns in a clockwise direction. *Ans.* $\frac{+7}{+1}$ or $+\frac{7}{1}$

36. Make a complete tabulation of the number of revolutions of each link and find the angular velocity ratio between the gear C and the arm A of the epicyclic train of Fig. 167. Let the fixed gear B and gear C have the same numbers of teeth as in Problem *35*. Assume that arm A turns in a clockwise direction. *Ans.* $\frac{-5}{+1}$ or $-\frac{5}{1}$

37. In an epicyclic gear train like that of Fig. 169 the annular E is the stationary link and the gears have numbers of teeth as follows: $t_b = 28$, $t_c = 30$, $t_d = 26$, $t_e = 140$. The arm is caused to rotate in a clockwise direction at 60 r.p.m.

(*a*) Present a complete tabulated solution.

(*b*) Obtain the angular velocity ratio of gear B to arm A and properly interpret it.

(*c*) Find the r.p.m. of gear *B*.

Ans. (*b*) $\frac{-4}{+1} = -4$ (*c*) 240 r.p.m.; counterclockwise

38. The driving shaft of the reverted epicyclic train of Fig. 170 makes 150 revolutions per minute and turns in a clockwise direction. The numbers of teeth of the gears are as follows: $t_b = 40$, $t_c = 41$, $t_d = 40$, and $t_e = 41$.

(*a*) Make a complete tabulated solution.

(*b*) Find the angular velocity ratio of the driven shaft to the drive shaft (or of gear *E* to arm *A*).

(*c*) In what direction does the driven shaft rotate?

(*d*) What is the r.p.m. of the driven shaft?

Ans. (*b*) $+\frac{81}{1681}$ (*d*) 7.23 r.p.m.

CHAPTER X

CAMS

Introduction. The design of many machines includes certain links to which various more or less irregular and properly timed motions must be given. Included among such machines are various types of prime movers, together with an almost unlimited number of so-called "automatic" machines such as screw machines, gear-cutting machines, printing machinery, textile machinery, etc. The necessity of producing these irregular motions in as simple a manner as possible has led to the use of direct-contact mechanisms called *Cam Mechanisms.*

A cam mechanism may be defined as one which can transform, by direct contact, one of the simple motions (generally uniform rotation but occasionally oscillation or reciprocation) into a definite, prescribed, irregular motion. In fact it might be stated that almost any direct-contact mechanism which gives a somewhat variable motion to its follower is a cam mechanism; obviously there must be a great variety. Some of the more common forms are discussed in this chapter.

Links of a Cam Mechanism. Any cam mechanism consists in the main of three links; namely,

1. The driving link or *Cam*
2. The driven link or *Follower*
3. The fixed link, which supports the other members and constrains their motion, the *Frame*

Cams. In its most usual form the cam is either a plate or a cylinder which imparts a prescribed motion to its follower by means of the peculiar outline of its working surface or by means of a groove cut into its surface. See Figs. 172 and 173. It generally receives its motion by being fastened to a constant-speed driving shaft, which of course gives to it a uniform rotation.

The cam shown in Fig. 172 is commonly referred to as a *Plate* or *Disk Cam.* Its outline (the profile of its working surface) is in contact with a *Roller Follower* which in this case is permitted by the guides in the frame to move only along the vertical center line of the cam. In a single revolution of the cam about the axis of the *Cam Shaft,* the follower is caused to move upward a distance equal to its *Total Displacement* and then to return to its initial position. On its upward stroke, the follower is under the complete influence (constrainment) of the cam. On the downward stroke, however, this is not true and the follower must be held in contact with the cam either by the action of gravity or by a spring which was previously compressed during the upward (or outer) stroke. Disk cams may be designed to work with other kinds of followers than the roller follower. They may be used whenever a follower is required to move in a plane which is perpendicular to the axis of the cam shaft.

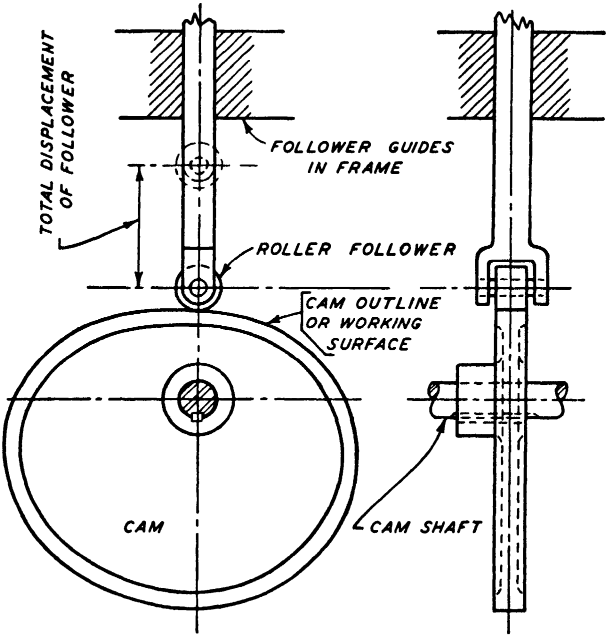

TOTAL DISPLACEMENT OF FOLLOWER
FOLLOWER GUIDES IN FRAME
ROLLER FOLLOWER
CAM OUTLINE OR WORKING SURFACE
CAM
CAM SHAFT

Fig. 172 Disk Cam

Some cams, which are either very similar to disk cams or are actually disk cams, are known as *Positive Return Cams* due to the fact that they are in full control of their followers, not only on the outward stroke but also on the return stroke. Two cams of this type are shown in Figs. 187(*a*), 187(*b*), and 188.

That type of cam which is cylindrical in form and causes motion of its follower by means of a groove cut into its surface is called a *Cylindrical Cam.* Such a cam is shown in Fig. 173, carrying in its groove a *Conical Follower.* The latter will have clearance with the bottom of the groove as well as with one side of the groove. The groove is shaped to give to the follower its prescribed motion as it moves through its total displacement and then returns to the initial position shown in the figure. The path of the follower is dependent upon the manner in which it is constrained by the frame, and may be a straight line (parallel to the axis of the cam) or a curved line; however, either of these lines will lie in a plane tangent to the cylinder and therefore parallel to its axis. A cycle of some cylindrical cams consumes more than one revolution. The follower of a cylindrical cam is under the control of the cam at all times.

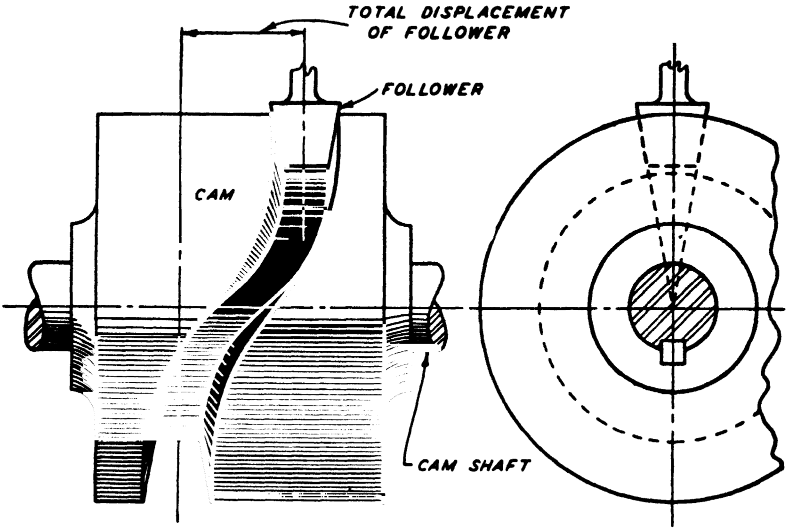


Fig. 173. Cylindrical Cam

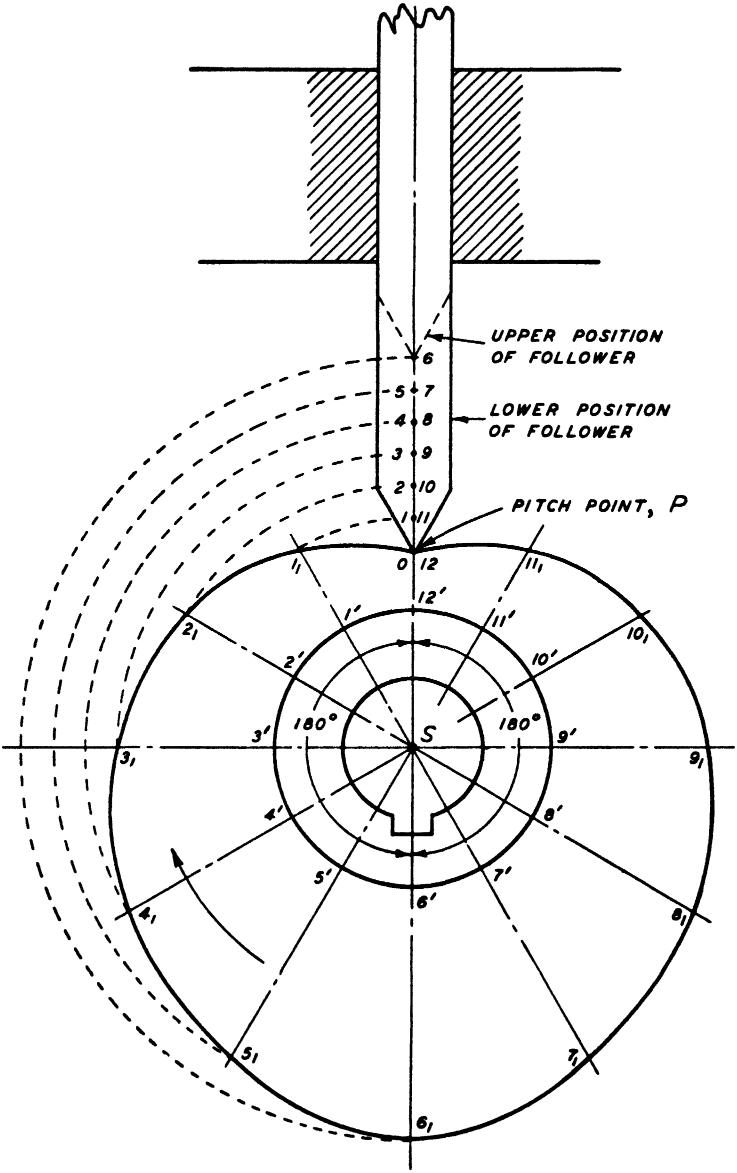
The design of the cam consists in laying out the profile of its surface or the location of its groove so that the cam can transform its uniform rotation into the exact prescribed motion of the follower. In so doing, the cam must not only give to its follower a certain type of motion, but it must also cause the follower to reach a definite position at a certain time or after a certain time interval. Equal time intervals may be measured by equal angular displacements of the cam, since the cam is given a uniform rotation. The student should give particular attention to the preceding statement for it is fundamental to the layout of all cams. Many factors influence the design of a cam; these become evident in the several layouts that are given farther on.

Followers. The follower is the driven link of the mechanism. Its path is generally a straight line or an arc of a circle. Its motion may be continuous or intermittent, uniform or variable. Its action with the cam may be either rolling or sliding. The name given a follower is often dictated by its form, as is evident from the following: knife-edge or wedge-shaped follower, roller follower, conical follower, mushroom or flat-face follower, etc. On the other hand, the name assigned to the follower may be dictated by its path or motion, such as: radial follower, offset follower, oscillating follower, etc. The form, path, initial position, and type of motion of the follower are all influencing factors in the layout of a cam. In some cam mechanisms, the motion of the follower is fully constrained by the cam, while in others it is only partially constrained.

Frames. The frame is the supporting link of the cam mechanism. It provides bearings for the cam shaft, holds the pivot for an oscillating follower, and provides guides for a translating follower.

Fundamental Procedure in a Disk Cam Design. Although the design of each disk cam differs to some extent from the others, there is still a great similarity in the methods by which these designs are carried out. The solution of the several examples introduced in this chapter should serve to acquaint one with the ideas by which cam designs in general can be analyzed and effected.

Example. The first example is the design of the cam of Fig. 174. In this example the cam is to rotate in a clockwise direction. The follower is to be of the knife-edge type, and its path is coincident with the vertical center line of the cam. The initial position of the follower is given as 1 inch above the cam center. The cam, while rotating through a half turn (180 degrees) is to cause the follower to rise a distance equal to 1 inch, taking the several successive equidistant positions (indicated by the numbers *0* to *6*) in equal intervals of time. During the second half-turn and the completion of the cycle, the cam is to return the follower (or permit the follower to be lowered) to its



SCALE, FULL SIZE

Fig. 174

initial position, through the successive equidistant positions from *6* to *12* in equal intervals of time.

Solution. At any convenient point locate *S*, the center of the cam, and through it draw the vertical and horizontal center lines. Locate the end of the follower at its initial position, *P* (or *0* and *12*), so that $SP = 1$ inch. Lay off a distance of 1 inch upward from *0* (zero) locating point *6*, the end of the follower's travel or stroke. Divide the distance *0-6* into 6 equal divisions and number them in order from *0* to *6* as shown. Since the follower is to occupy these same points on its return stroke, number these points in the reverse direction with numbers from *6* to *12*.

Now the follower is to occupy, in succession, the 6 equidistant positions *1*, *2*, *3*, *4*, *5*, and *6* in equal intervals of time, and the total time consumed in the complete upward stroke, *0* to *6*, is the time taken by the cam in rotating through that central angle of 180 degrees (or one half turn) which is turning upward toward the follower. Evidently, then, if this central angle of 180 degrees is divided into 6 equal angles, each of the latter divisions represents the equal time interval or unit consumed while the follower is being raised through $\frac{1}{6}$ of its stroke. Hence, by the radial lines *1′*, *2′*, *3′*, *4′*, *5′*, and *6′*, divide the angle of 180 degrees (measured against the rotation of the cam) into these 6 equal divisions. (This division can be accomplished by the use of a protractor, or standard triangles, or by describing any circle with *S* as a center and dividing its half-circumference with the dividers into 6 equal arcs by the "cut-and-try" process.) Since the cam rotates clockwise in this design, the follower must be raised to position *1* while the radial line *1′* is rotating into the vertical position. Likewise, while *2′* is rotating into the vertical center line, the follower must take position 2; and so on. Hence there must be points of the cam profile on these radii, *1′* and *2′*, which are at the same radial distance from the cam center as points *1* and *2* respectively. Therefore with *S* as a center and with radii equal to the distances *S1*, *S2*, *S3*, *S4*, *S5*, and *S6*, describe arcs (against the direction of rotation) intersecting *1′*, *2′*, *3′*, *4′*, *5′*, and *6′*, at 1_1, 2_1, 3_1, 4_1, 5_1, and 6_1 respectively. Making use of the French curve, draw a smooth curve through these 6 points to obtain this part (one-half in this case) of the cam profile.

In a similar manner the other 180 degrees (or the central angle through which the cam turns in lowering the follower from *6* to *12*) is now divided into 6 equal parts by the radial lines *7′*, *8′*, *9′*, *10′*, *11′*, *12′*. With S as a center and radii $S7$, $S8$, etc., describe arcs which intersect their respective radii at 7_1, 8_1, etc. Connect these points to finish the cam profile.

Cam Follower Motions. In order to simplify the cam design of Fig. 174, the type of motion of the follower was given indirectly by providing the several positions along its path which it was to take after equal time intervals had elapsed. Since these positions were equally spaced, the follower was caused by the cam to move through equal distances in equal units of time. Hence the follower in the cam mechanism of Fig. 174 was given a uniform motion by this assignment of positions along its path. As a rule, however, these positions are not assigned, but instead a certain type of motion is given to the follower and it is necessary at the beginning of the design of its cam to locate the several positions along its path. Such positions along the path, or divisions of the follower's displacement, depend upon the character of the motion.

Referring again to Fig. 174 it will be seen that, with uniform motion, the follower leaves its initial position at a velocity which it retains to the end of its stroke. In other words, its velocity is constant and its acceleration (change in velocity) is equal to zero. This is liable to introduce into the follower (and hence into the other links of the machine to which the follower may be attached) a considerable shock at the beginning and end of the motion. This shock may be of such a magnitude that it is very undesirable. It may be relieved or practically eliminated by the substitution, when possible, of another type of motion which would permit the follower to gradually increase its velocity at the beginning and gradually decrease its velocity toward the end of its stroke. For this reason, as well as others, several different types of follower motions have been evolved.

The requisite characteristics of the motion of a cam follower are obtained in most cases from one or more of the following motions:

1. *Uniform Motion,* designated by the letters *U.M.*
2. *Simple Harmonic Motion,* designated by the letters *S.H.M.*
3. *Uniformly Accelerated and Retarded Motion,* designated by the letters *U.A.R.M.*

Uniform Motion. When a follower has uniform motion, the positions along its path after equal intervals of time may be obtained in the following manner. Referring to Fig. 175, let line *AB* represent the displacement of the follower. Draw the indefinite line *AC* at any acute angle with *AB*. Having decided upon the number of divisions into which *AB* is to be divided, assume any distance such as *0-1′* and lay this distance off on line *AC* as many times as it is required to divide the given line *AB*. Number these divisions from *0* to *8′*, since 8 divisions were taken in this figure. Draw a line from point *8′* to point *B* (or *8*), the end of line *AB*. Now through each point on *AC* draw a line parallel to *8-8′*, intersecting *AB* as shown. Then *0-1*, *1-2*, *2-3*, etc., are the 8 equal distances through which the follower will be moved in 8 equal time intervals, or equal angular displacements of the cam, to give to the follower a uniform motion.

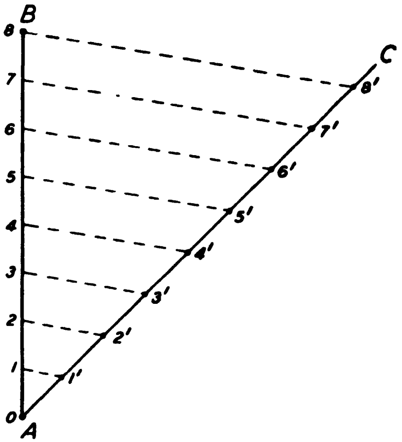


Fig 175

It should be noted that the number of divisions of the follower's displacement is also the number of points which will be located to determine the profile of the cam. This is evident from an inspection of Fig. 174. The correctness of the profile depends of course upon the number of points by which it is determined. The points should be located closely enough to each other to leave no doubt as to the direction of the profile at any time. Hence consideration should be given to this when deciding upon the number of divisions into which the displacement of the follower is to be divided. It follows that for the same degree of accuracy, fewer points on the profile are needed for a small angular turn of the cam than for a large one.

Simple Harmonic Motion. With this type of motion the velocity of the follower is increased from zero at the initial position of the follower to a maximum at the mid-position, and is then decreased to zero at the final position. The change in velocity, or the acceleration, is a maximum at the initial position of the follower. It decreases to zero at its mid-position and continues to decrease, providing (during the second half of the follower's travel) a negative acceleration (or a deceleration) which becomes a maximum at the final position. It is evident then that this motion is one which relieves the tendency to shock at the beginning and end of the follower's displacement.

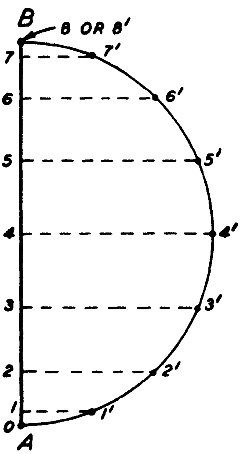


Fig 176

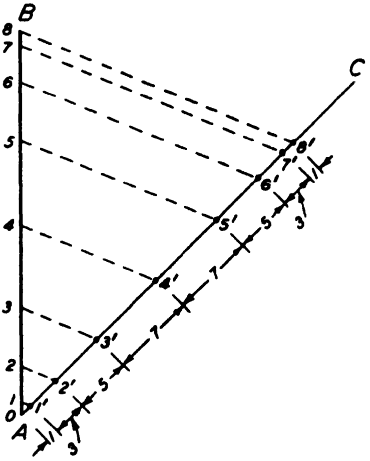


Fig 177

The method by which the path of the follower is divided in order to secure the positions which the follower with simple harmonic motion will take after equal intervals of time is as follows: Referring to Fig. 176, describe a semicircle which has the given displacement *AB* for its diameter. Divide the semicircle into a number of equal divisions, say 8 divisions as in the figure. Number the points thus obtained with numbers from *0* to *8′* as shown. Draw perpendiculars from these points to the diameter, obtaining points *0, 1, 2, 3,* etc. These points are the required positions of the follower, and the unequal distances between them are traveled by the follower in equal intervals of time.

Uniformly Accelerated and Retarded Motion. The location of the various positions of the follower for this type of motion will be described with the help of Fig. 177. Again line *AB* is given as the displacement of the follower. From point *A* draw the indefinite line *AC*, making any acute angle with line *AB*. Assume some small unit of length and lay it off from point *A* on *AC*, as the distance *0-1′* which locates point *1′*. Locate point *2′* on *AC* so that the distance *1′-2′* is equal to three of these small units or three times the distance *0-1′* as shown by the dimension *3*.

In a similar manner, lay off points *3′* to *8′*, in order, so that each point is distant from the one which precedes it by the number of units shown in the corresponding dimension of the figure. Then the divisions assumed on the line *AC* from *0* to *8′* will be proportional to the numbers *1, 3, 5, 7, 7, 5, 3, 1*. (If 6 divisions are used, the divisions will be proportional to the numbers 1, 3, 5, 5, 3, 1.) Next, connect point *8′* on line *AC* with point *B* (which is also named *8*). Then through each of the other points on *AC* draw lines parallel to line *8-8′*, obtaining intersections with *AB* at points *1, 2, 3,* etc. The latter are the required positions of the follower, and the distances between them are the distances the follower must be caused to move in equal intervals of time, and hence in equal angular displacements of the cam, if its motion is to be uniformly accelerated and retarded along its path.

As the name implies, with uniformly accelerated and retarded motion the velocity of the follower is increased from zero at the initial position to a maximum at the mid-position, from which point the velocity is uniformly decreased to zero at the final position. Thus during the first half of its displacement the follower has a uniform acceleration, while during the second half it has a uniform deceleration or retardation.

Disk Cam with Roller Follower. A disk cam having a roller follower is shown in Fig. 178. The follower rotates on a pin which is carried by the follower arm, and it is caused to move along the vertical center line of the cam. In the case of a knife-edge follower (see Fig. 174), the pitch point of the follower is taken at the tip or edge of the follower, but with a roller the pitch point is taken at the center of the roller. Then the design of the cam with a roller follower is exactly the same procedure that would be followed for a knife-edge follower having an edge located at the center of the roller. The outline thus obtained is that of the so-called *Pitch Surface* of the cam. See Fig. 178. From the pitch surface, the *Working Surface* of the cam with a roller follower is obtained. The working surface, in the case of either follower, is that surface of the cam which directly contacts the surface of its follower. Therefore, with a knife-edge follower, the pitch surface and the working surface have coincident outlines or profiles.

Example. The manner in which the working surface for a roller follower is obtained will be demonstrated in the following description of the design of Fig. 178.

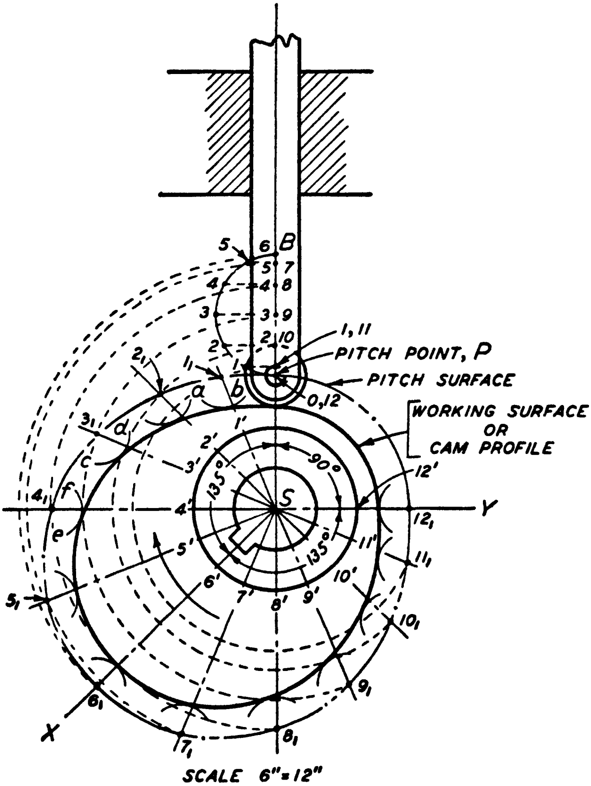


Fig. 178. Disk Cam with Roller Follower

(Note that Fig. 178 is shown drawn to scale, half size.)

In the latter the following information is given:

Rotation of cam, clockwise

Type of follower, roller

Path of follower, along vertical center line of cam

Diameter of follower, 5/8 inch

Distance from center of cam to center of roller at initial position, 1 3/8 inches

Follower advances 1¼ inches in ⅜ turn (135 degrees) of cam with S.H.M.

Follower returns 1¼ inches (to initial position) in ⅜ turn of cam with S.H.M.

Follower rests during ¼ turn (90 degrees) of cam.

Solution. At any convenient point, locate center *S* and through it draw the horizontal and vertical center lines of the cam. Locate point *P*, the center of the roller follower, at a distance of 1⅜ inches above *S*. Lay off the stroke or total advance, *PB*, 1¼ inches upward from point *P*. Since this total advance is made with simple harmonic motion (S.H.M.), describe a semicircle on *PB* as a diameter. Divide the semicircle into, say, 6 equal divisions indicated by the numbers *0* to *6*. Project these points by perpendicular projectors upon *PB* to obtain points similarly numbered *0* to *6*. Then points *1*, *2*, *3*, *4*, *5*, and *6* on the path of the follower are the positions to which the center of the follower will be successively advanced by the cam after equal intervals of time.

Since the total advance is made in ⅜ of a turn of the cam in a clockwise direction, lay off angle *PSX* (equal to ⅜ of 360 degrees, or 135 degrees) from line *SP* in a direction opposite to, or against, the direction of rotation of the cam. Divide angle *PSX* into the same number of equal parts as the advance, *PB*, obtaining radial lines *1′*, *2′*, *3′*, *4′*, *5′*, and *6′*. With *S* as a center and radii equal to *S1*, *S2*, *S3*, *S4*, *S5*, and *S6*, describe arcs intersecting radial lines *1′*, *2′*, etc. at points 1_1, 2_1, 3_1, 4_1, 5_1, and 6_1 respectively. These points are now connected to give the outline of the pitch surface of the cam through the first 135 degrees, during which the follower is raised from *P* to *B*.

Since the follower returns from *B* to its initial position, *P*, with S.H.M., the same points that were used in its advance may also be used on its return. Number these for the return with numbers *6* to *12* inclusive.

Lay off from line *SX*, and against the direction of rotation, the central angle *XSY*, equal to 135 degrees (which is the angular displacement of the cam during the return stroke). Divide angle *XSY* into 6 equal parts by the radial lines *7′* to *12′*. (Note that the direction in which they are numbered is again opposite to the direction of the cam's rotation.) With *S* as a center and radii *S7*, *S8*, etc., describe arcs intersecting radial lines *7′*, *8′*, etc., at points 7_1, 8_1, 9_1, 10_1, 11_1, and *12*$_1$ respectively. These points are now connected by a smooth curve to give the outline of the pitch surface through the second angle of 135 degrees, during which the follower is returned from *B* to its initial position *P*.

Since the follower is to remain at rest and hence stay at its initial position for the rest of the cycle, all points on the pitch surface must be at a distance *SP* (or *S-12*$_1$) from the center *S*. Therefore, complete the pitch surface by describing a circular arc from *12*$_1$ to *P*, with *S* as a center and *S-12*$_1$ as a radius. (Note that this pitch surface which is now completed would be the cam profile, or the working surface, if the cam were to be used with a knife-edge follower.)

With the completion of the pitch surface, set the compass to a radius equal to that of the follower, or $^5/_{16}$ inch, and with centers taken at many points along the complete outline of the pitch surface describe circular arcs such as *ab*, *cd*, *ef*, etc. With the French curve draw a smooth curve tangent to these arcs. This is the cam profile or working surface.

Example. Given data:

(Note that Fig. 179 is shown drawn to scale, half size.)

Rotation of cam, counterclockwise

Type of follower, roller

Path of follower, along vertical center line of cam

Diameter of follower, $^5/_8$ inch

Distance from center of cam to center of roller at initial position, $1^7/_{16}$ inches

Follower advances $^3/_4$ inch in $^1/_4$ turn (90 degrees) of cam with S.H.M.

Follower rests during $^1/_{12}$ turn (30 degrees) of cam

Follower advances $^3/_4$ inch in $^1/_4$ turn (90 degrees) of cam with S.H.M.

Follower returns $1\frac{1}{2}$ inches (to initial position) in $^1/_3$ turn (120 degrees) of cam with U.A.R.M.

Follower rests during $^1/_{12}$ turn (30 degrees) of cam.

Solution. Locate center *S* and draw horizontal and vertical center lines of cam. Lay off *SP* equal to $1^7/_{16}$ inches to locate *P*, the pitch point or initial position of center of roller. Lay off distance *PA* equal to $^3/_4$ inch and locate points *0*, *1*, *2*, *3*, and *4* thereon in accordance with S.H.M. Lay off distance *AB* equal also to $^3/_4$ inch and locate points (numbered for this motion also) *0, 1, 2, 3,* and *4* thereon for S.H.M. Divide distance *BP,* the return stroke, into six divisions which are proportional to the numbers 1, 3, 5, 5, 3, 1, in accordance with the return motion which is U.A.R.M. Number the points thus located on *BP* by the numbers *0, 1, 2, 3, 4, 5* and *6.*

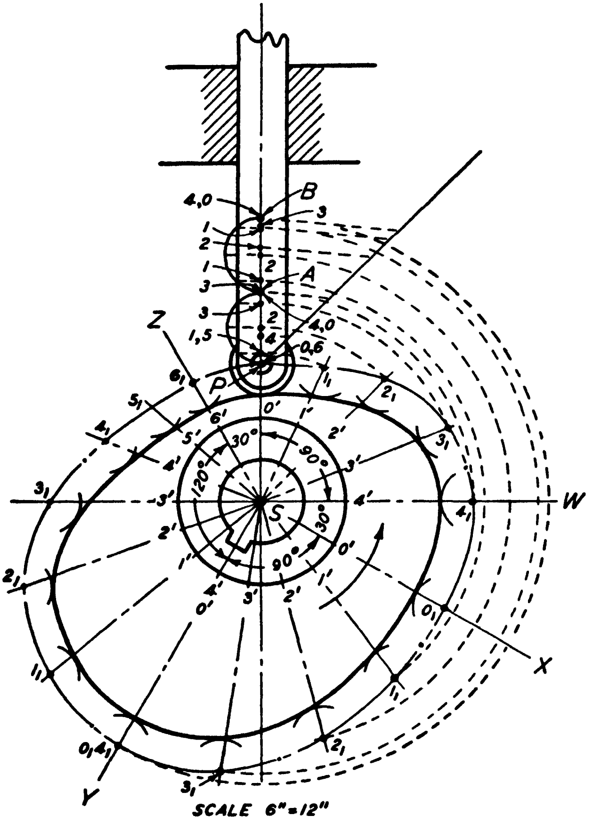


Fig. 179

Lay off angle *PSW* equal to 90 degrees from line *SP* (against the direction of rotation of the cam) and divide it into 4 equal parts to obtain the radial lines *1', 2', 3',* and *4'.* Lay off angle *WSX* equal to 30 degrees and number this radial line *0'.* During this angle the follower rests between its two upward advances. Lay off angle *XSY* equal to 90 degrees and divide it into 4 equal parts to obtain the radial lines numbered again by *1', 2', 3'* and *4'.* Lay off angle *YSZ* equal to 120 degrees, which is the time interval for the return stroke, and divide it into 6 equal parts to obtain the radial lines marked by the numbers *1'*, *2'*, *3'*, *4'*, *5'* and *6'*. The angle *ZSP*, equal to 30 degrees, is the angle (or time interval) through which the cam turns while the follower stays or rests at its initial position.

As in the previous examples, circular arcs can now be described with *S* as a center and distances from *S* to the several points on *PB* (taken in order upward and then in order downward) as radii. The intersection of each of these arcs with the corresponding radial line of the proper angular displacement is a point on the outline of the pitch surface. Connect these points by a curve which includes the circular arcs representing the rest periods through angles *WSX* and *ZSP*. The outline thus obtained is that of the pitch surface.

The cam profile, or outline of the working surface, can be obtained by the method which was given in the preceding example.

Disk Cam with Flat-Face Follower. It is often expedient in cam mechanisms to employ a follower of the type used in Fig. 180. Due to the plane surface it affords for contact with the working surface of the cam, it is known as a *Flat-Face Follower.* As in the case of other followers, flat-face followers may be given either a reciprocating or an oscillating motion.

The cam mechanism of Fig. 180 is typical of this sort of design. It should first be noted that the plane of the follower's face is at an angle of 90 degrees with the stem of the follower; the axis of the stem intersects the axis of the cam; and the follower is caused to reciprocate along its stem and hence along a radial line (the vertical center line) of the cam. Otherwise, the statement of the example is as follows:

(Note that Fig. 180 is shown to scale, half size.)

Rotation of cam, clockwise

Distance from face of follower at initial position to center of cam, 1⅜ inches

Follower advances 1¼ inches in ⅜ turn (135 degrees) of cam with U.A.R.M.

Follower rests during ¼ turn (90 degrees) of cam.

Follower returns 1¼ inches (to initial position) in ⅜ turn of cam with U.A.R.M.

Solution. Locate center *S* and through it draw the vertical and horizontal center lines of the cam. Locate point *P* on the vertical center line at a distance of $1\frac{3}{8}$ inches above *S*. Lay off the stroke of the follower, *PB*, equal to $1\frac{1}{4}$ inches. Divide *PB* into any even number of divisions, say 6, according to the given follower motion which is uniformly accelerated and retarded. Number the points thus obtained on *PB* by numbers from *0* to *6* inclusive. Since the return motion is also U.A.R.M. the same divisions and their numbers can be used in the reverse order on the return stroke.

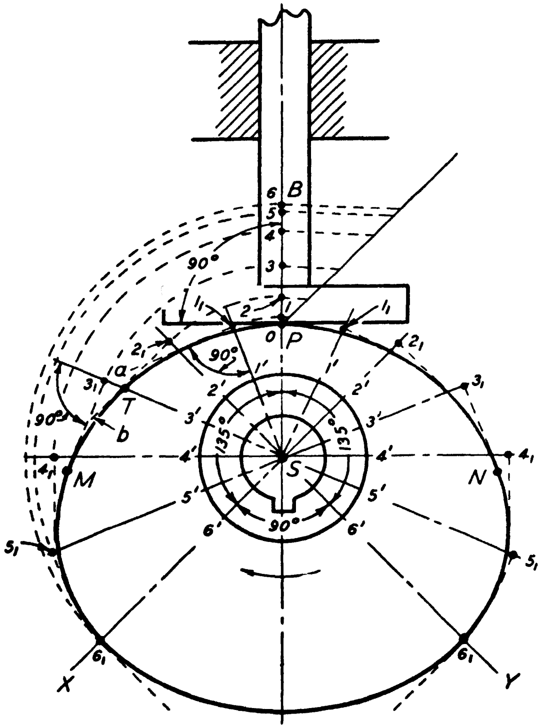


Fig. 180

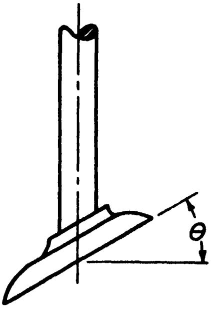


Fig. 181

Lay off central angle *PSX*, equal to 135 degrees, against the rotation of the cam and divide it into the 6 equal angles (or equal time units) indicated by the radial lines *1'*, *2'*, *3'*, *4'*, *5'*, and *6'*. Next lay off angle *XSY*, equal to 90 degrees, for the rest period of the follower. There remains, at the center, the angle *YSP* equal to 135 degrees. Divide the latter into 6 equal parts indicated by the radial lines, in order, *6'*, *5'*, *4'*, *3'*, *2'*, and *1'*.

With *S* as a center and radii *S1*, *S2*, etc., describe arcs intersecting the corresponding radial lines at points 1_1, 2_1, 3_1, etc., of the first angle of 135 degrees. With *S* as a center and the same radii used in reverse order, obtain intersections with radial lines *6′* to *1′* of the second angle of 135 degrees at points 6_1, 5_1, etc. At each of these points of intersection with a radial line, draw a line perpendicular to the radial line, obtaining such perpendiculars as 1_1a, 2_1b, etc.

Draw a smooth curve tangent to these perpendiculars. This curve is the cam profile, or outline of the working surface of the cam, which will give to the flat-face follower its required motion. It should be noted that the cam profile which lies between the two points marked 6_1 is in action during the rest period and therefore is a circular arc of radius *S6*.

It is evident that each perpendicular drawn to a radial line of the cam is in reality a position of the face of the follower. For example, after the cam has rotated through the angle 2_1SP, the radial line *2′* will coincide with the vertical center line; point 2_1 will be at point *2*; and the perpendicular 2_1b will be the position of the face of the follower. At this instant the follower will be in contact with, and tangent to, the cam at the new position to which point *T* (the point of tangency of 2_1b and the cam profile) has been rotated. Therefore the longest perpendicular (from its point of tangency with the cam profile to the radial line with which it is perpendicular) is the minimum length of the follower face to the right or left of its center line. In this layout, the longest distances are equal to each other, due to symmetry, and are 3_1N on the right and 3_1M on the left side. The follower face must for practical reasons, extend a trifle further than these respective distances to the right and left of the vertical center line of the cam.

Although, in this example of Fig. 180, the axis of the stem passes through the axis of the cam, this cam outline as designed would afford the same motion to a flat-face follower having an axis parallel to the given position of this example.

It should also be mentioned that the face of a flat-face follower may be at some other angle than 90 degrees with the axis of the stem of the follower. This condition is illustrated in Fig. 181, in which the face of the follower makes an acute angle, θ, with the axis of the stem. In designing a cam to take this type of flat-face follower, the procedure is the same as that used in the design of the cam of Fig. 180, with the single exception that face angle, θ, is used in the place of 90 degrees when drawing the lines 1_1a, 2_1b, etc.

Mushroom Follower. The two-view drawing of Fig. 182 shows a cam with a reciprocating flat-face follower similar to that of Fig. 180.

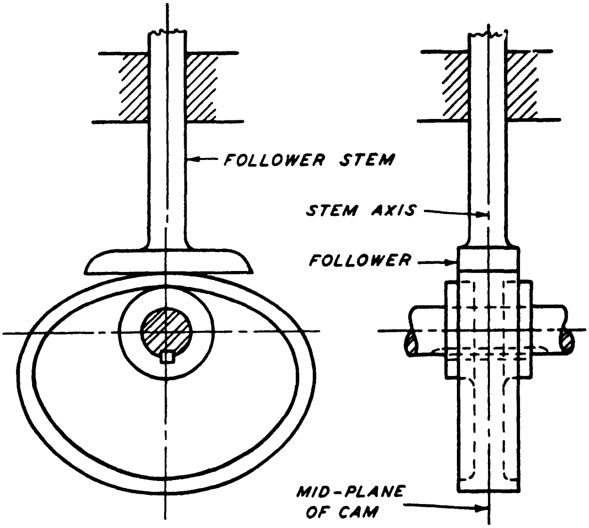


Fig 182

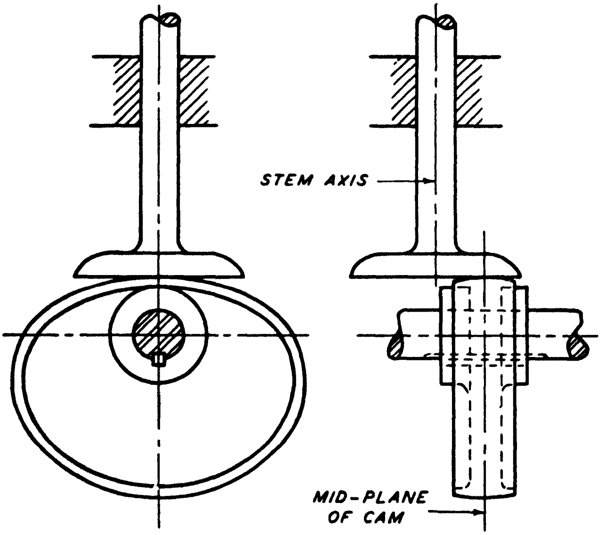


Fig. 183

It will be noted that the axis of the follower stem in Fig. 182 not only intersects the cam axis but also lies in the mid-plane of the cam. The cam of Fig. 182 is also shown in Fig. 183. In the latter, however, it can be seen that while the axis of the follower stem still intersects the axis of the cam, it no longer lies in the mid-plane of the cam, but is offset therefrom. In both of these positions the cam affords the same reciprocating motion to the follower. However, in Fig. 183 the offset position permits the cam to give to the follower, as the latter moves along its stem, a rotary motion about the axis of its stem. In order to be rotated, it is obvious that the stem of the follower must be cylindrical and the face of the follower must be circular in form. Such a flat-face follower, which is caused to simultaneously translate along, and rotate about, the axis of its stem, is known as a *Mushroom Follower*. Its rotation facilitates lubrication and permits the distribution of wear over a larger area of the face of the follower.

Disk Cam with Offset Roller Follower. In each of the preceding examples dealing with disk cams having roller followers, the follower has a reciprocating motion along a path coinciding with the vertical center line of the cam. In this example the follower again has a reciprocating motion, but its path is along a line which is parallel to the vertical center line of the cam and hence offset as shown in Figs. 184(*a*) and 184(*b*). Although the construction in this case uses mainly the principles previously introduced, the student should note carefully the few new ideas which are involved.

Example. The given information for the design is as follows:

(Note that Figs. 184(*a*) and 184(*b*) are drawn to scale, half size.)

Rotation of cam, clockwise

Type of follower, roller

Initial position of follower (center), 1 inch to the left of, and 1¼ inches above, the center of the cam shaft

Path of follower, a right line parallel to the vertical center line of cam

Diameter of follower, ⅝ inch

Follower advances 1¼ inches in ¼ turn (90 degrees) of cam with S.H.M.

Follower rests during ¼ turn (90 degrees) of cam.

Follower returns 1¼ inches (to initial position) in ½ turn (180 degrees) of cam with U.A.R.M.

Solution. Refer to Figs. 184(*a*) and 184(*b*). The latter figure is given in order to show the location of the initial position of the follower and to present some work which if shown on Fig. 184(*a*) would make that figure too complicated.

Locate the cam center S, the horizontal and vertical center lines of the cam, the initial position P of the center of the follower, and the path of the follower, PB. Secure positions of the follower during its rise along PB in accordance with the assigned follower motion.

Extend PB through the horizontal center line of the cam and, with S as a center, describe a circle tangent to this line at point A. This circle will be referred to as *Circle A*. It plays an important part in this layout because any line drawn tangent to it will at some instant during the rotation of the cam coincide with the follower path, PB. Also, radius SA of this circle is the reference line from which the start is made to lay off (against the direction of rotation) the angular displacements of the cam as outlined in the assigned motion.

Lay off the central angle ASX equal to 90 degrees, the angular displacement during the rise of the follower. The arc of circle A that is intercepted by angle ASX is next divided into 6 equal parts, or the same number of divisions previously obtained for the simple harmonic motion of the follower. Give the points thus obtained numbers from $1'$ to $6'$ inclusive, and at these points draw tangents to circle A, extending them in the direction of rotation of the cam.

Now with S as a center and $S1$, $S2$, etc., as radii, describe arcs which intersect in order the tangents previously drawn, at 1_1, 2_1, 3_1, 4_1, 5_1, and 6_1. (See Fig. 184(*b*) for point 3_1.) Connect these points by a smooth curve to obtain the pitch line of the cam through the first ¼ turn.

Lay off the central angle XSY and place the number $6'$ at the end of its intercepted arc on circle A. This arc has the same number at each end because the follower stays at point 6 throughout this angular displacement. Draw a line $6'B'$ tangent to circle A at this new point $6'$. With S as a center describe an arc with radius $S6$ from 6_1 on tangent $6'6_1$, to its intersection (also called 6_1) with tangent $6'B'$. This arc is the continuation of the pitch line during the second ¼ turn of the cam, during which the follower is resting.

The positions of the follower for its return during the final ½ turn of the cam, or the remaining central angle YSA, can now be

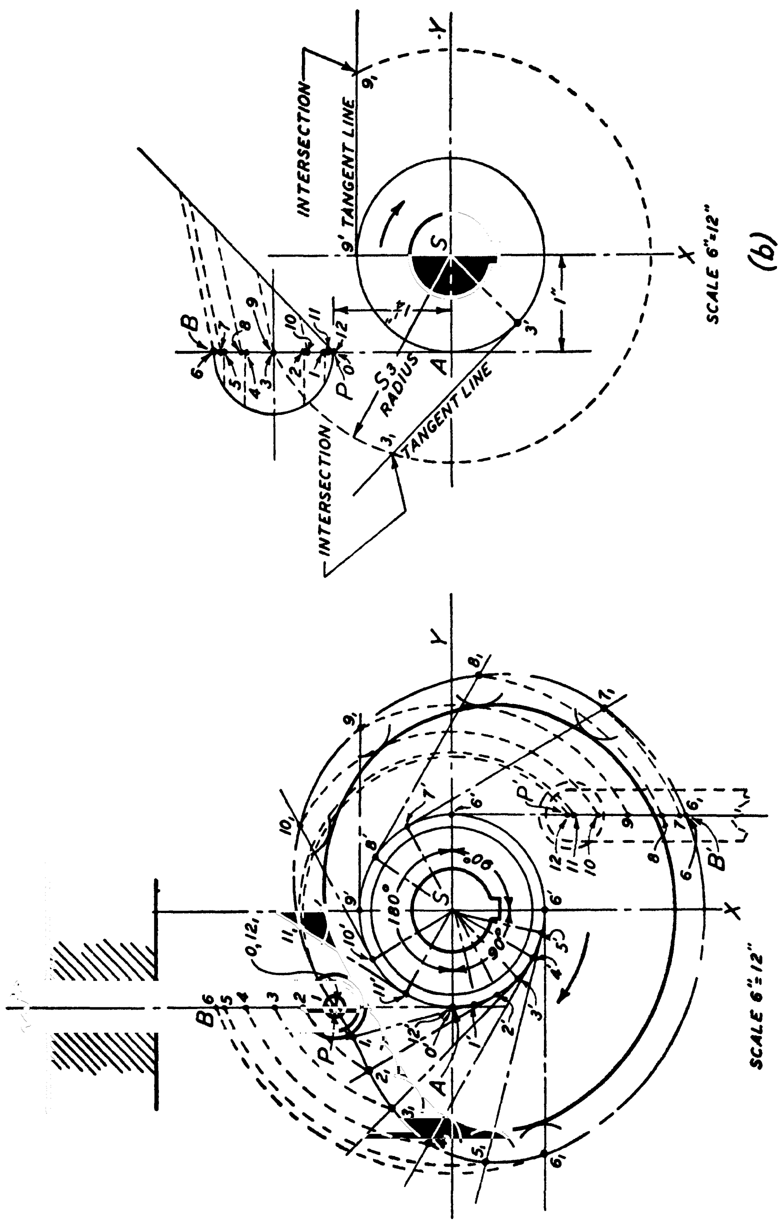

INTERSECTION
9' TANGENT LINE
9,
TANGENT LINE
INTERSECTION
3,
3'
S3 RADIUS
S
A
B
P
X
Y
1"
SCALE 6"=12"
(b)
180°
90°
B'
P'
SCALE 6"=12"
(a)

Fig 184

obtained for the given motion, either on *PB* directly (as shown in Fig. 184[*b*]) or on *P'B'* of Fig. 184(*a*). The latter is in reality a revolved position of *PB*, secured by rotating *PB* to the initial tangent *6'B'* of the final angular displacement.

Divide the arc of circle *A* included within angle *YSA* into the divisions as indicated by points *7'* to *12'* inclusive. Draw tangents to circle *A* at these points and, with *S* as a center and radii *S7*, *S8*, etc.,

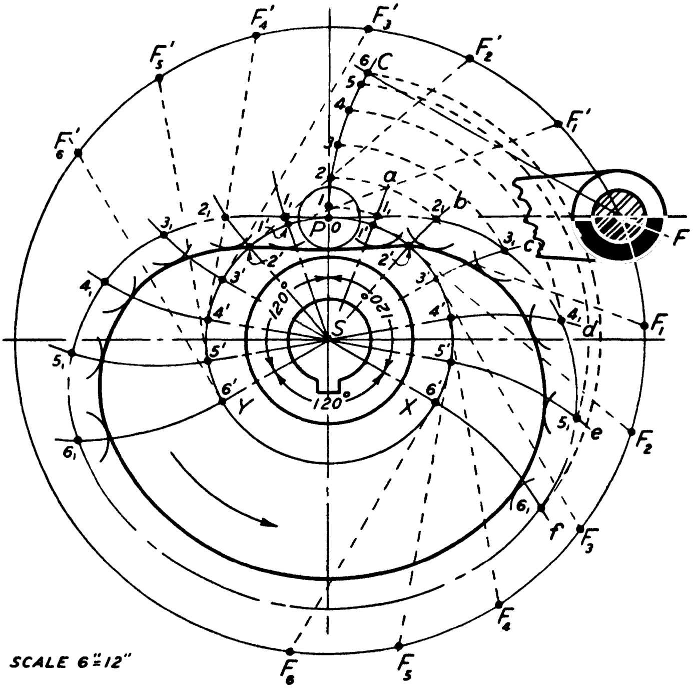


Fig. 185(*a*)

describe arcs intersecting (in order) the tangents to *7'*, *8'*, etc., at points 7_1, 8_1, 9_1, 10_1, 11_1, and 12_1. (See Fig. 184(*b*) for intersection 9_1.) Connect these points to complete the pitch line.

Obtain the cam profile to work with the roller follower by the method used in previous examples.

Disk Cam with Oscillating Roller Follower. All the cams discussed so far, had followers with points which all moved along straight lines. In other words each follower had a motion of rectilinear translation, reciprocating back and forth within the limits of its path.

The cam to be investigated in this article is one having a follower which oscillates; that is, each point of the follower moves back and forth in the arc of a circle. This is brought about by attaching the follower to an arm which is connected to the frame by a pivot or short pin whose axis is fixed. The follower in this case is a roller pinned to the end of the arm so that, as it oscillates with the arm, it also rotates about its axis relative to the arm.

Example. The following information is given for this example:

(Note that Figs. 185(a) and 185(b) are shown drawn to scale, half size.)

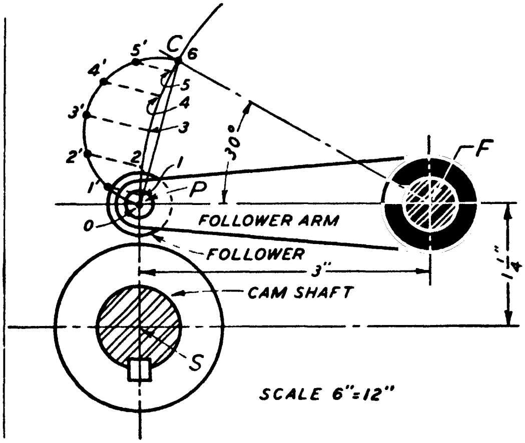


Fig. 185(b)

Rotation of cam, counterclockwise

Type of follower, pivoted or oscillating roller

Diameter of follower, ⅝ inch

Length of follower arm, 3 inches

Distance from center of cam to center of roller at initial position, 1¼ inches

Distance from center of pivot to horizontal center line of cam, 1¼ inches

Total angular displacement of roller, 30 degrees

Follower advances through arc of 30 degrees in ⅓ turn (120 degrees) of cam with S.H.M.

Follower rests during ⅓ turn of cam.

Follower returns to initial position in ⅓ turn of cam with S.H.M.

Solution. Refer to Figs. 185(*a*) and 185(*b*). The latter figure is given in order to show the work more clearly by relieving the main drawing of some of its construction.

Locate the cam center *S*, the arm center *F*, and the follower center *P*. Draw all necessary center lines.

With *F* as a center and *FP* as a radius, describe arc *PC*, the path of the follower through 30 degrees. Locate the follower positions for the advance from *P* to *C* by dividing the arc *PC* according to the given simple harmonic motion. (This can be done in most cases with sufficient accuracy by using the chord *PC* as the diameter of the semicircle as shown in Fig. 185(*b*). Otherwise the arc must be rectified and used as the diameter of the semicircle and the divisions obtained laid out upon the arc.) Number these positions with numbers from *0* to *6*, inclusive, and note that the same positions can be used in the reverse order on the return of the follower to its initial position.

With *S* as a center and *SP* as a radius, describe a circle referred to hereafter as *Circle P*. Lay off the angular displacements of the cam, angles *PSX*, *XSY*, and *YSP*, each equal to 120 degrees as given.

Secure points *1′*, *2′*, *3′*, *4′*, *5′*, and *6′* by dividing that arc of circle *P* which is intercepted by angle *PSX* into 6 equal parts, or time intervals, to correspond with the divisions taken on *PC*. (This angle *PSX* is the first angular displacement considered, for in all cams the construction is always carried on against the direction of rotation of the cam.)

With *S* as a center, describe a circle through point *F* and intersect this circle (in order) at F_1 to F_6 inclusive by tangents drawn to circle *P* at points *1′* to *6′* inclusive. With points F_1 to F_6, inclusive, as centers, and a radius *FP* (3 in.) draw the indefinite arcs *a*, *b*, *c*, *d*, *e*, and *f* extending outward from circle *P* at points *1′* to *6′* respectively.

Now with *S* as a center and radii *S1*, *S2*, *S3*, *S4*, *S5*, and *S6*, describe arcs intersecting arcs *a*, *b*, *c*, *d*, *e*, and *f* at points 1_1, 2_1, 3_1, 4_1, 5_1, and 6_1 respectively. These points determine that part of the pitch line of the cam which functions during the advance of the follower during the first 120 degrees of the cycle.

The pitch line, during the 120-degree rest of the follower at the end of its advance, will of course be a circular arc whose radius is equal to *S6*.

The same procedure as that used during the first angular displacement of 120 degrees to obtain points 1_1 to 6_1 inclusive may now be used during the last angular displacement of 120 degrees to locate points 6_1 to 1_1 inclusive. The complete construction is given in the figure.

Having obtained the pitch line, the cam profile can be obtained therefrom in exactly the same manner as in the preceding examples dealing with roller followers.

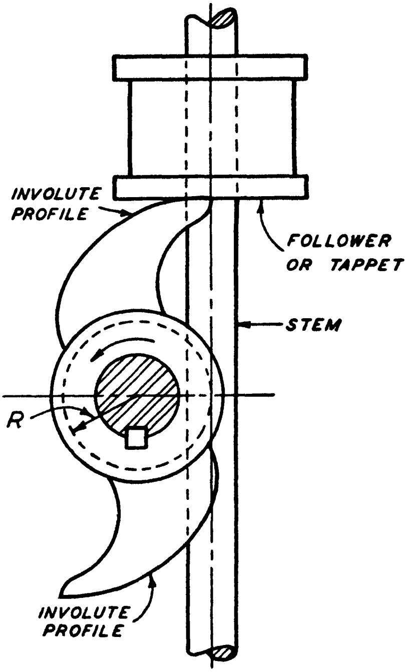


Fig. 186(a). Involute Cam

It is evident that the shape of the follower arm of the cam of Fig. 185 makes it necessary to offset the arm from the plane of the cam in order to avoid interference. The arm however could be given another shape which would provide clearance with the cam throughout its cycle and thereby make it possible for the follower to operate in the mid-plane of the cam.

Involute Cam. The *Involute Cam* receives its name from the involute curve which is used as the outline of its working surface. It is a disk cam that works with a reciprocating flat-face follower. The follower is raised along the axis of its stem to a point at which further rotation of the cam causes the follower to be released, whereupon it drops to its initial position, from which it is again picked up and raised by the cam. The axis of the stem is generally located in such a manner that it is tangent to the base circle or evolute upon which the involute profile is formed. This is the circle whose radius is R in Fig. 186(*a*). These cams are generally of the double involute type shown in Fig. 186(*a*). Obviously in this figure the follower is raised twice during each revolution.

This cam finds application in the fine crushing of ores in what is known as a *Stamp Mill.* Hence it is often referred to as a *Stamp Mill Cam.* In this application the follower or tappet is attached to a cylindrical stem. To the lower end of the stem is fitted securely a cast iron "head" or "boss" which is provided on its under side with a steel "shoe." The stem, boss, and shoe constitute the "stamp" which may weigh as much as 1,000 pounds. As the stamp is allowed to drop, ore is caught and crushed between its shoe and a die which is fixed at the bottom of the mortar into which the ore is fed. The action of the cam on the tappet not only raises the stamp but also imparts to it a slight rotary motion about its axis. This rotary motion is caused by the sliding of the cam upon the plane surface of the tappet, and it prevents the stamp from having successive identical contacts with the die. As a rule, in order to secure the desired capacity, several stamps are used in a single crushing unit. In such a case the battery of cams is arranged along the cam shaft in a manner that will give the desired sequence of action between each stamp and its die. See Fig. 186(*b*).

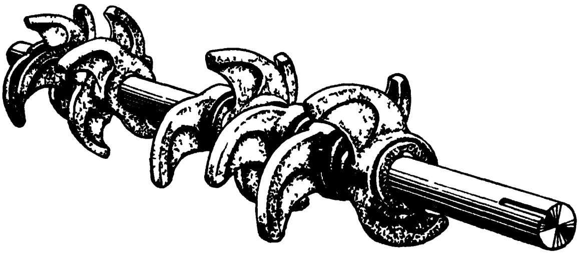


Fig 186(*b*) Stamp Mill Cam Shaft Assembly

Positive Motion Cams. So far in this chapter no follower discussed has been under the control of its cam on the return stroke. Gravity, or a mechanical means such as the use of a spring which is compressed on the outward or advance stroke, was resorted to in each of these cases to hold the follower to its cam as it returned to its initial position. Hence the follower was not positively driven by its cam throughout the cycle. In certain installations, however, it is either feasible or absolutely necessary that the cam be so designed as to be in full control of its follower throughout the cycle. Such a cam is known as a *Positive Motion* or *Positive Return Cam.*

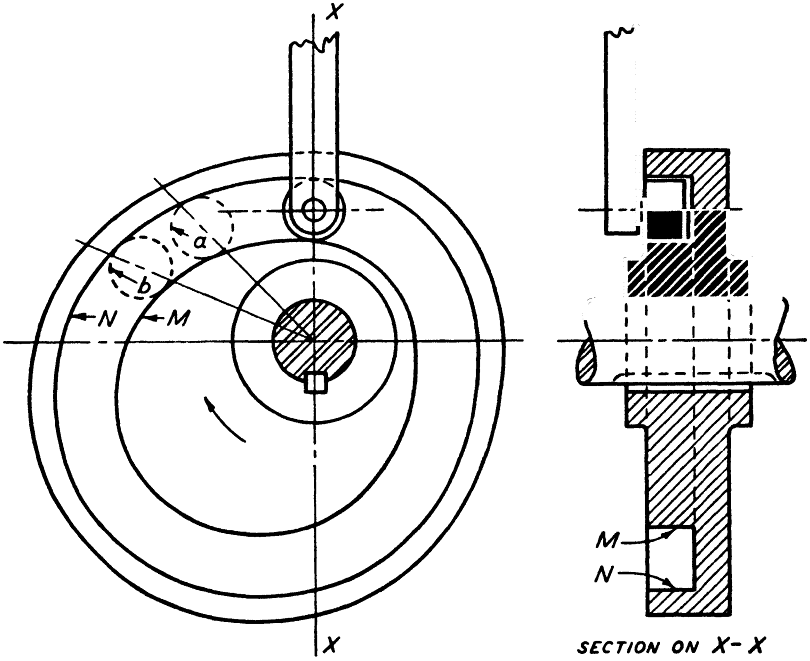


Fig. 187(*a*). Face Cam, a Positive Motion Cam

Face Cam. One type of positive-motion cam is the so-called *Face Cam* of Fig. 187(*a*). A roller follower riding in the groove of this cam is forced upward or outward along a radial line of the cam through contact with the inner surface of the groove, and is pulled inward or returned to its initial position through contact with the outer surface of the groove. It is evident that the follower can be placed in different positions relative to the cam, just so long as the guides in the frame constrain its movement to a radial line of the cam; for instance, it might be placed on the right side of the cam where it would be caused to reciprocate along the horizontal center line.

Contact between the follower and the inner surface of the groove is seen by the figure to produce counterclockwise rotation of the follower about its own axis, while contact with the outer surface of the groove produces clockwise rotation of the follower. Hence the latter must reverse its direction of rotation twice per revolution of the cam. This reversal of motion is an unfavorable characteristic, for some slippage must take place, and excessive wear at points along the contact surfaces may result.

The design of this cam is practically the same as that of any disk cam which is to take a reciprocating radial roller follower. The inner surface, M, is laid out in exactly the same way as the working surface of a disk cam. The outline of the outer surface, N, is then obtained by describing circles such as a and b of Fig. 187(a), with a radius equal to that of the roller, tangent to the outline of the inner surface throughout its extent. A curve drawn tangent to these circles is the outline of the outer surface. If these circles are drawn with a radius equal to that of the follower, the latter will just fit into the groove, and the roller will contact both groove surfaces at the same time. In order to avoid this, a small amount of clearance can be provided by making the radius of such circles as a and b a trifle larger than that of the roller.

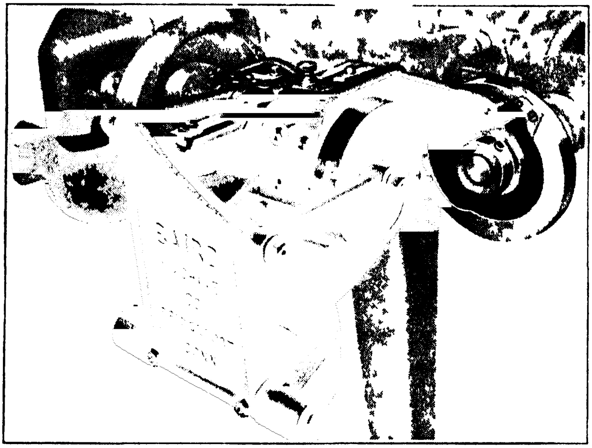


Fig 187(b) Positive Motion Cam
Courtesy of Baird Machine Co Bridgeport Conn

For the purpose of comparison, the cam of Fig. 187(a) is designed on the basis of the same given data that was used for the cam of Fig. 178. Comparison of these figures will show that the inner surface, *M*, of Fig. 187(*a*) and the working surface of Fig. 178 are identical.

An application of a positive motion cam of this type is illustrated in Fig. 187(*b*).

Constant-Breadth Cam. Another positive-motion or positive-return cam is the *Constant-Breadth Cam* of Fig. 188. This cam is designed in such a manner that it is always tangent to two parallel plane surfaces which are maintained at a constant distance apart by being internal surfaces of a reciprocating link which surrounds the cam. The distance between these parallel surfaces is the constant breadth (dimension K of Fig. 188) maintained by the cam, and from which it gets its name. The motion for which this type of cam may be designed can be prescribed for only a half turn of the cam; motion for the other half turn is necessarily the same as that for the first, but it is carried out in the reverse order.

Thus it will be seen from the design, (which follows) of the cam of Fig. 188, that the working surface *PCA*, while contacting the upper surface of the follower, permits the follower to rest for a certain time interval and then raises it with simple harmonic motion a distance equal to *PB*. While doing this, the cam has rotated through 180 degrees, which brings point *P* of surface *PCA* into contact with the lower surface at point *A*. The action from there on is between the working surface *PCA* and the lower surface of the follower, so that *PCA* must start the second half turn by letting the follower rest at the top of its stroke for the same time interval that it previously rested at the bottom of its stroke; it then returns with simple harmonic motion to its initial position.

The given information for the design of the cam of Fig. 188 is as follows:

(Note that Fig. 188 is shown to scale, half size.)

Rotation of cam, clockwise

Distance from center of cam to upper surface of follower at initial position, 1¼ inches

Follower rests during 30 degrees of rotation of cam.

Follower advances 1½ inches during 150 degrees of rotation of cam with S.H.M.

Locate center *S* and draw the horizontal and vertical center lines of the cam. Locate point *P* on the vertical center line 1¼ inches above

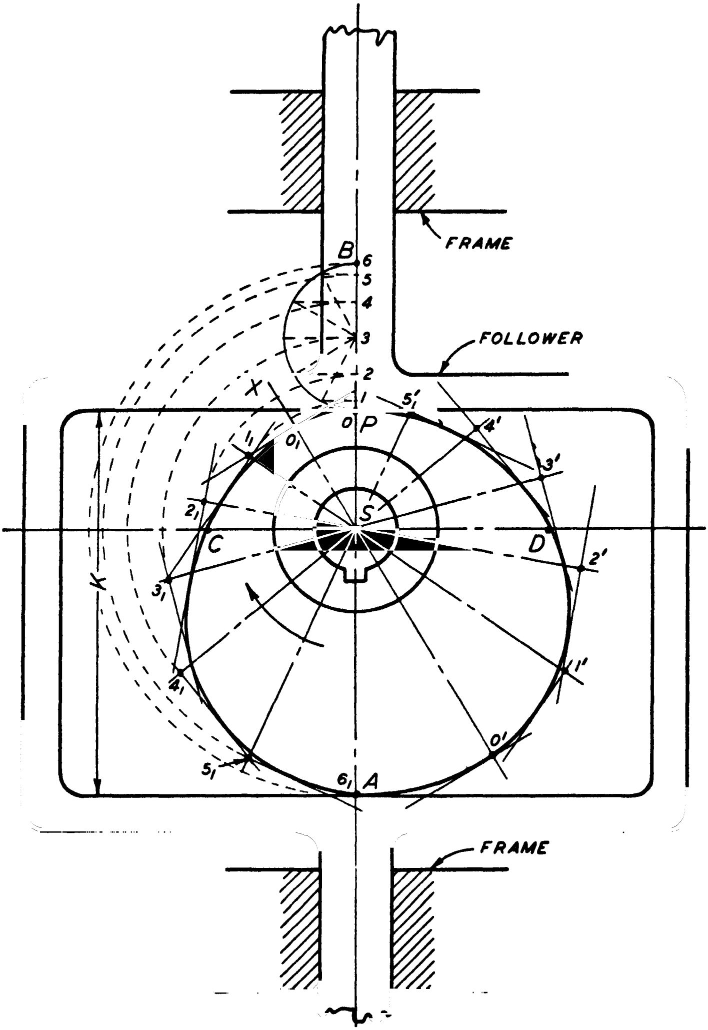


Fig. 188. Constant-Breadth Cam, a Positive Motion Cam

S and draw a horizontal line through it to represent the upper surface of the follower. Lay off *PB* equal to 1½ inches, the follower displacement, and divide it into 6 divisions numbered *0* to *6*, in accordance with simple harmonic motion.

Lay out a 30° central angle, PSX, against the rotation of the cam, for the time interval during which the follower stays at its initial position. With S as a center and SP as a radius, describe an arc intersecting radius SX at 0_1, thus obtaining arc $P0_1$ as part of the cam profile.

Divide angle 0_1SA (150 degrees) into 6 equal parts and finish the cam profile from point 0_1 to point A by the method used in the example of Fig. 180.

Next calculate the constant breadth, K, of the cam, which is evidently equal to twice the distance SP plus the stroke of the follower. Therefore

$$K=PA=2\times SP+PB$$
$$=2\times1\tfrac{1}{4}+1\tfrac{1}{2}=4 \text{ inches}$$

Extend the radial lines 0_1S, 1_1S, 2_1S, etc., to the opposite side of the cam and lay off, on these radial lines, points $0'$, $1'$, $2'$, $3'$, $4'$, and $5'$ at a distance of 4 inches from 0_1, 1_1, 2_1, 3_1, 4_1, and 5_1, respectively. Through points $0'$ to $5'$, inclusive, draw lines perpendicular to their respective cam radii. Draw a curve tangent to these perpendiculars to secure the remainder of the cam profile.

It should be noted that by this construction, the cam profile is everywhere tangent to, and hence lies between, a pair of parallel lines which are at a distance from each other equal to K, the constant breadth of the cam.

General Concept of Cam Design. A portion of the working surface of the cam of Fig. 178 is reproduced in Fig. 189. This portion lies between the vertical center line and the radial line $4'$ of Fig. 178 and is denoted by ef in Fig. 189. In the latter let the cam be fixed or stationary and let the follower be rotated about center S against the given direction of rotation of the cam until its center line coincides with the radial line $3'$. The follower of course rolls on the cam surface as it rotates about S, so that the center of the roller is advanced outward along its center line a distance equal to 0-3, taking the new position 3_1, and giving to the follower as a whole the new position shown. Now let the cam and follower be rotated together about center S in a clockwise direction until radial line $3'$ coincides with the vertical center line of the cam. This will cause them to take the dotted position in the figure with the roller center at point 3 and the cam profile at e_1f_1. This dotted position is, however, the actual relationship of cam to follower that is produced by the regular operation of the cam mechanism through the same amount of rotation of the cam alone. Therefore the position of the follower at 3_1, with respect to the cam profile, is identical to that at point 3 along the actual path of the follower. When this is noted in the light of the preceding cam examples, it will be seen that, in reality, points on the pitch line of each cam have been determined by holding the cam in a fixed position and rotating the follower against the assigned rotation of the cam to the various positions it must occupy after certain intervals of time have elapsed.

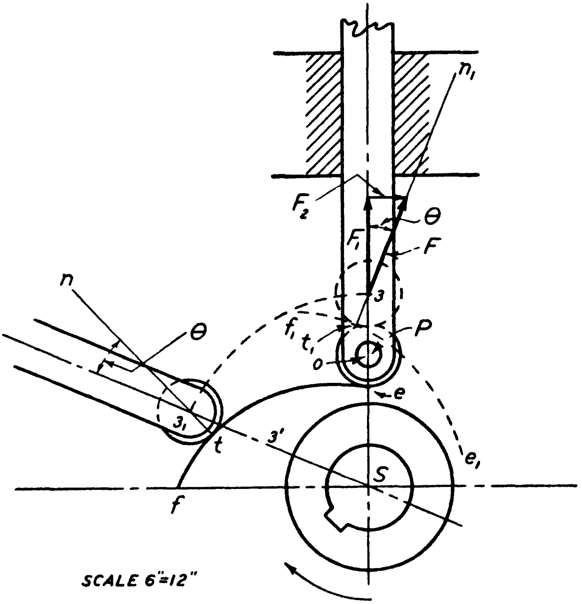


Fig 189

Pressure Angle of a Cam. Since a cam mechanism is a direct-contact mechanism, the force, with which the cam as the driver actuates the follower, is directed along the common normal which is the line of action of the mechanism. The common normal, as shown in the chapter dealing with the fundamentals of direct-contact mechanisms, is the line drawn through the point of contact of the driver and its follower perpendicular to their common tangent. The common normal at the phase shown in Fig. 189 is the indefinite line

tn or t_1n_1, and the force (or pressure) between the cam and its follower at this phase is directed along this line.

The angle between the common normal and the axis of the follower is called the *Pressure Angle* of the cam. The pressure angle, at the phase shown in the figure, is the angle θ. The magnitude of this angle is dependent upon the type of motion of the follower, its accompanying angular displacement, and the size of the cam. For most cams, throughout the successive phases of a cam cycle, the pressure angle is highly variable in magnitude, and in many designs it must be kept within a certain maximum value. This is because of its effect or influence upon the components of the pressure along, and perpendicular to, the axis of the follower. This can be shown in conjunction with Fig. 189, in which the normal pressure, F, is seen to have two components, F_1 and F_2. F_1, directed along the axis of the follower, is the useful component in that it is the only component that compels the follower to move. F_2, directed at right angles to the follower, sets up a pressure against the surfaces of the guides, which may result in too much friction and its accompanying wear along the contacting surfaces. It may be seen that, with a given normal pressure, the larger the pressure angle, the smaller the useful actuating component F_1, and the larger the destructive (or at least useless) component F_2 becomes. Hence a small maximum pressure angle is desirable, particularly under severe working conditions.

Variation of Pressure Angle with Size of Cam. In Fig. 190 two alternative cam outlines, a_1b_1 and a_2b_2, are shown. These will give identical motions to a roller-follower through the same angular displacement of the cam. The profile a_1b_1 is obtained with the initial position of the follower at P_1, while a_2b_2 is obtained with the follower initially at P_2, which is further removed from the cam center S than P_1 and so necessitates the use of a larger cam.

With the cam assuming the profile a_1b_1, the follower at a certain phase will have the relative position *1* with respect to the cam, and the common normal will be t_1n_1, so that θ_1 is the pressure angle. With the cam assuming the profile a_2b_2, the follower (at the same phase as before) will take the relative position *2*, and its common normal will be t_2n_2, so that θ_2 becomes the pressure angle. It is evident from the figure that θ_2 is less than θ_1. Therefore, increasing the size of a cam (by taking the initial position of the follower at a greater distance from the center of the cam shaft) decreases the pressure angle at all phases throughout the cycle and hence decreases the pressure angle at that phase where it is a maximum.

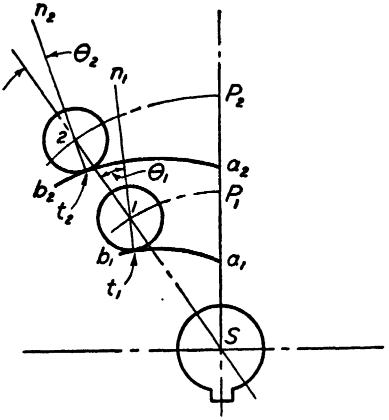


Fig. 190

Cam Chart or Displacement Diagram. In the solution or investigation of a cam design it is often convenient and helpful to use that which is known as a *Cam Chart* or *Displacement Diagram.* As shown in Fig. 191, such a diagram is a displacement-time graph or curve which represents the motion of the follower through the successive units of time of the complete cam cycle. The axes which are used in plotting this graph are: a vertical axis along which follower displacements are taken, and a horizontal axis along which time units or cam displacements are taken. It follows then that the abscissa of any point on the curve is a follower displacement and the ordinate is the corresponding cam displacement.

The method of laying out such a diagram is described in relation to Fig. 191, which is the displacement diagram of the example of Fig. 178. Note that it, like Fig. 178, is shown to scale, half size. In the latter the motion is given as follows:

Follower advances 1¼ inches in ⅜ turn of cam with S.H.M.
Follower returns 1¼ inches in ⅜ turn of cam with S.H.M.
Follower rests during ¼ turn of cam.

Construction. Lay out the vertical and horizontal axes intersecting at point *P′*. Lay off on the vertical axis a distance *P′B′* equal to 1¼ inches, the follower displacement.

Take any distance such as $P'C$ along the horizontal axis to represent the time consumed in one revolution (or cycle) of the cam. Since the cam has a constant speed, distance $P'C$ also represents one complete turn, or an angular displacement of 360 degrees. Divide $P'C$ into parts which are proportional to the angles involved in the motion; namely, 135, 135, and 90 degrees, which make the total of 360 degrees.

Divide $P'B'$ into 6 parts in accordance with the prescribed simple harmonic motion, numbering them from *0* to *6* as shown. Since the return of the follower is the same motion as the advance, these points may also represent the positions of the follower on the return stroke and may be numbered in order from top to bottom from *6* to *12* inclusive.

In agreement with the division of $P'B'$, divide each of the angular displacements of 135 degrees into 6 equal parts and number them from *1'* to *12'* inclusive. Each time unit thus produced is evidently $\frac{1}{6}$ of 135 degrees or $22\frac{1}{2}$ degrees. This leaves 4 such equal units of time, numbered from *13'* to *16'* inclusive, in the 90-degree period of rest.

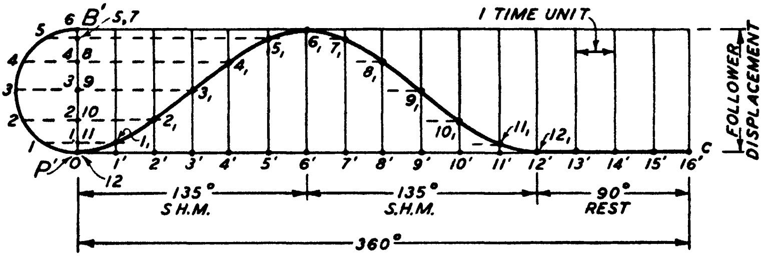


Fig 191. Displacement Diagram

Draw vertical lines at points *1'*, *2'*, *3'*, etc., and intersect these lines, in order, by horizontal lines drawn from points on $P'B$ obtaining intersections 1_1, 2_1, 3_1, etc., as shown. Connect these points by a smooth curve for the displacement-time graph.

Comparison of Fig. 191 with Fig. 178 will show that the two are similarly numbered, and this indicates the relationship that exists. Note in particular that the vertical lines *1'*, *2'*, *3'*, etc., of Fig. 191, have their counterpart in radial lines *1'*, *2'*, *3'*, etc., of Fig. 178.

Fig. 192 (which is drawn to scale, half size) is the displacement diagram of the cam mechanism of Fig. 179 which has a motion as follows:

Follower advances $\frac{3}{4}$ inch in $\frac{1}{4}$ turn of cam with S.H.M.

Follower rests during $\frac{1}{12}$ turn of cam.

Follower advances $\frac{3}{4}$ inch in $\frac{1}{4}$ turn of cam with S.H.M.

Follower returns $1\frac{1}{2}$ inches (to initial position) in $\frac{1}{3}$ turn of cam with U.A.R.M.

Follower rests during $\frac{1}{12}$ turn of cam.

The student should study the construction of Fig. 192 carefully in comparison to that of Fig. 179. They are similarly numbered to facilitate the comparison. Notice the different time intervals along the time, or horizontal, axis. This is often necessary in order to secure the desired number of points to determine the curves for the different motions in the successive angular displacements.

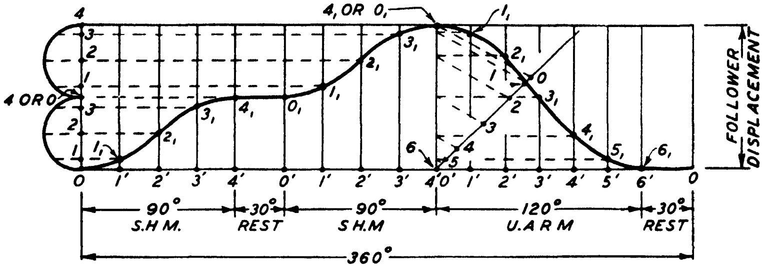


Fig. 192. Displacement Diagram

Displacement Diagram in Design. When the size of a cam is dictated by the space that it may occupy or by the distance from the initial position of the follower to the center of the cam shaft, the displacement diagram need play no direct part in the design of the cam. This is evident from the several designs which have been included in this chapter, in none of which was a diagram introduced to effect the construction. However, when the size of a cam is dictated by the limitation of the size of its pressure angles to a certain maximum, the displacement diagram is of great use and plays an important part in the design, particularly in the case of disk cams with roller or knife-edge followers. The utility of a displacement diagram is due to the relationship which its length, *P'C* of Fig. 191, may assume to the cam design and to its ability to disclose (at least very closely) the pressure angles, and hence the location of the maximum pressure angle which will occur if the design of the cam is based upon the diagram. To provide a cam whose maximum pressure angle is exactly equal to a prescribed limiting value requires considerable mathematical involvement. Hence it will be the aim of this text to provide a method by which a cam can be designed so that it will not exceed the prescribed maximum value.

When the design of a cam is based on the displacement diagram, the length of the diagram becomes the circumference of a circle which is known as the base circle of the cam. In this manner the length of the diagram directly affects, and becomes related to, the size of the cam. The base circle of a cam is a circle that passes through that point of the pitch surface of a cam at which the pressure angle is a maximum. Now the slope of the graph at any point on such a displacement diagram is very nearly equal to the pressure angle of that point when it becomes located on the cam. The slope of any curve at a given point is given by the angle between the tangent to the curve at that point and the horizontal axis, or by the angle between the normal to the curve and the vertical axis. It follows that the point of maximum slope has the maximum pressure angle. In the diagram of Fig. 191, the parts of the graph representing the advance and return strokes of the follower are identical. Therefore their maximum slopes, which are at their mid-points (points 3_1 and 9_1) are equal, and so the maximum pressure angle occurs at both of these points. This maximum pressure angle is the angle $m3_1n$ or θ at point 3_1 of Fig. 193(*a*). The latter is that part of the diagram of Fig. 191 which represents the first angular displacement of 135 degrees from point *0* to point *6′*.

For a given travel of the follower, the length of the displacement diagram affects the pressure angle at every point of the graph and hence defines the maximum pressure angle in whatever motion it may occur. This is evident in Figs. 193(*a*) and 193(*b*). In (*a*) the arbitrarily assumed length, *P′C* of the diagram, is as shown in Fig. 191, the distance *0-6′* along its base being 3/8 of *P′C* in length, due to the fact that 135 degrees is 3/8 of one turn, or 3/8 of 360 degrees. The maximum pressure angle, θ, created by the total length *P′C* of the diagram (and hence also by the proportionate part, *0-6′*) is found by the protractor to be around 40 degrees. It is now thought that, under the working conditions of the cam, this angle is too large and should therefore be materially reduced. Therefore, before proceeding with the design any further, another displacement diagram, having a considerably greater overall length than the first, is drawn in full. The latter is shown in part in Fig. 193(*b*), the part shown being the same as that in Fig. 193(*a*). The angle θ, upon being constructed in this new diagram, is

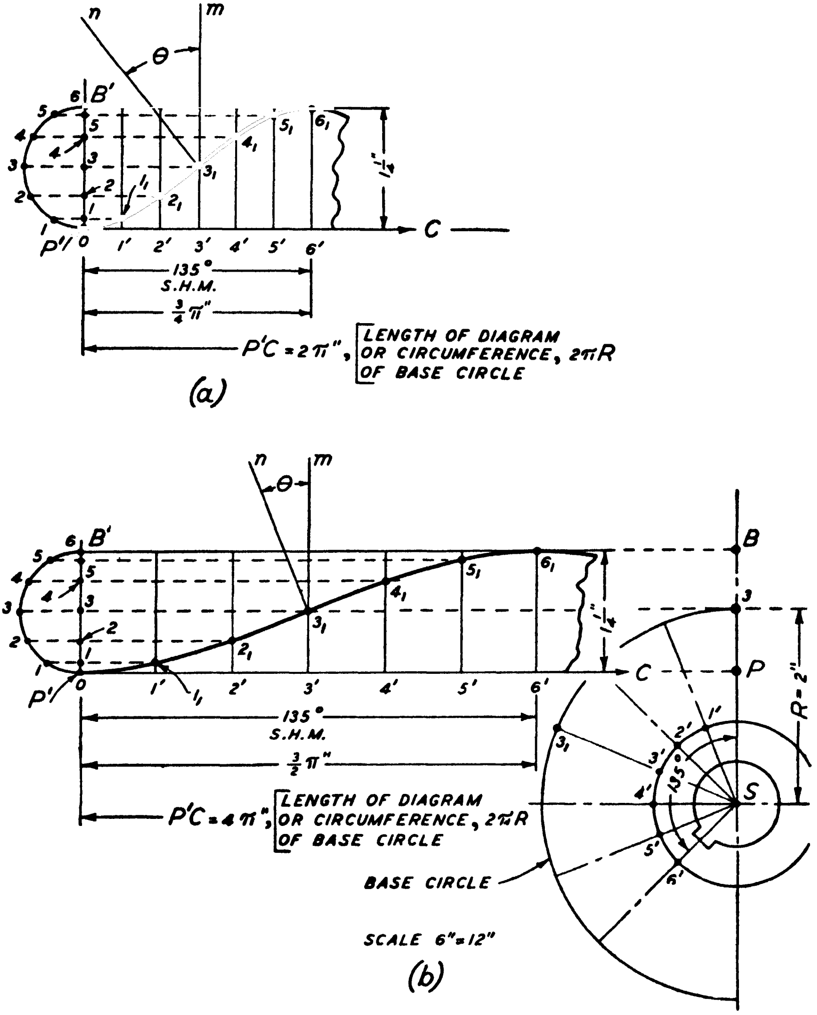


Fig. 193

found to be much smaller than in (*a*), due to the increased length of the diagram. Since it is well within the limit set for the maximum pressure angle, no further change in length is necessary, and the design of the cam is continued on this basis. Note that diagram (*b*) as constructed is 4π inches in length, or twice as long as 2π inches, the over-all length of (*a*). Therefore the distance *0-6'* in (*b*) is equal to

$$\tfrac{135}{360}\times 4\pi = \tfrac{3}{8}\times 4\pi = \tfrac{3}{2}\pi \text{ inches}$$

while the distance *0-6'* in (*a*) is equal to

$$\tfrac{135}{360}\times 2\pi = \tfrac{3}{8}\times 2\pi = \tfrac{3}{4}\pi \text{ inches}$$

Hence increasing the over-all length of the diagram increases proportionally the base lengths of the individual cam displacements of the diagram. It should be further noted that by proper mathematical procedure the length of the initial diagram could have been so calculated as to provide the desired pressure angle, thus eliminating the necessity to change the diagram. A more advance study of cams should cover this procedure.

Having secured a displacement diagram with a maximum pressure angle that is acceptable, it is next necessary to properly locate the center of the cam with respect to the initial position of the follower. (This is the time at which the displacement diagram functions directly in giving to the cam that size which will keep the maximum pressure angle within the set limit.) At one end of the diagram of Fig. 193(*b*) draw a vertical line and intersect it, as shown at point *3*, by a horizontal line drawn through that point of the diagram at which the pressure angle is a maximum. Since the length of the diagram is equal to the circumference of the base circle, divide this length by 2π to secure the base circle radius, *R*. (In the diagram as drawn, this would give $\frac{4\pi}{2\pi}$ or 2 inches as the base circle radius. Fig. 193(*b*) is to scale, half size.) Measure downward on the vertical center line a distance *R* to locate center *S* of the cam. Then the circle described with *S* as a center and *R* (*S3*) as a radius, is the base circle of the cam. Project point *P'* of the diagram upon the vertical center line to secure point *P*, the initial position of the follower. Project point *B'* of the diagram upon the vertical center line to secure point *B* and, therefore, the advance or stroke *PB* of the follower.

Since the motion of the follower was completely laid out on *P'B'* in securing the displacement diagram, these points can be projected across to *PB*, whereupon the cam outline can be procured in the usual manner. In doing this, points 1_1, 2_1, 3_1, etc., of the diagram take their respective positions on the radial lines of the cam which denote the angular displacements on the cam, in exactly the same manner as the vertical lines of the displacement diagram. Thus point 3_1 of the diagram on vertical line $3'$ will fall at point 3_1 at the intersection of radial line $3'$ and the base circle. It is at this point (and in this example at point 9_1 also) that the maximum pressure angle will occur.

Note. See Figs. 178, 189, and 193(*b*), all of which deal with the same example. The length of the diagram of Fig. 193(*b*) was purposely so chosen as to locate the initial point *P* of the cam at the same place as it was located by the given distance of 1⅜ inches in Fig 178, so that if the cam of Fig. 193(*b*) were completed it would be exactly the same as in Fig. 178.

Translation Cam. Sometimes the motion of a follower is produced by a cam which, instead of rotating about an axis, reciprocates with rectilinear motion. Such a cam is generally spoken of as a *Translation or Reciprocating Cam.* It is generally a block, as shown in Fig. 194, which rides in guides provided by the frame, the contacting face of the block being shaped to give the required motion to its follower. The motion of the follower is in a plane which is either perpendicular or parallel to the plane of motion of the cam. This type of cam may afford positive motion to a follower by carrying the latter in a groove which is either cut into the face of the cam or formed by straps attached to the face. In such a case the surfaces which contact the follower are of course shaped to provide the required motion.

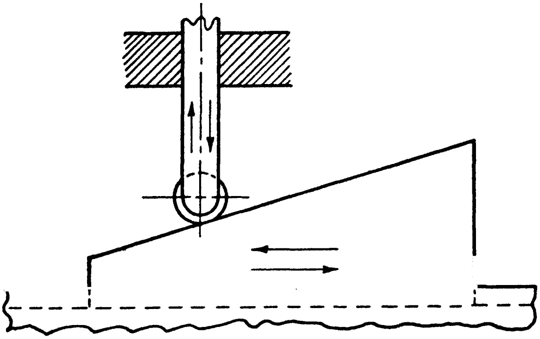


Fig. 194. Translation Cam

Cylindrical Cam with Reciprocating Follower. The design of a cylindrical cam, like that of any cam, depends upon the type of motion to be imparted to the follower, the constrainment of the follower's motion by the frame (and hence the path of the follower), and finally the type of follower to be used. While different kinds of followers are used with cylindrical cams, as a rule a conical follower is preferred unless some feature of the design makes it necessary or advisable to use another type. That conical follower which is used is always a frustrum of a cone whose apex is at the axis of the cam. This makes it possible for all of the points on the contacting element of the groove to tend to give to the follower the same rotative speed.

The drawing of the cylindrical cam generally consists of a projection drawing, together with the developments of the inner and outer edges or lines of the groove. The design of the cam may not include the complete drawing however, for the construction of the cam simply requires the layout of the outer edges of the groove upon a development of the surface of the cylinder. From this development a template can be made which, when wrapped around the cylinder, permits the outline of the groove to be transferred thereto. A milling cutter, of the size and shape of the follower, guided along the line so transferred, produces the groove in the cylinder.

Example. Obtain the projection and development of a cylindrical cam to be used with a reciprocating conical follower which moves along an element of the cam. Further information is as follows:

(Note that Fig. 195 is shown to scale, half size.)

Direction of rotation of cam, counterclockwise as viewed from lower end

Diameter of cam, 3 inches

Length of cam, 3¾ inches

Width of groove, ¾ inch

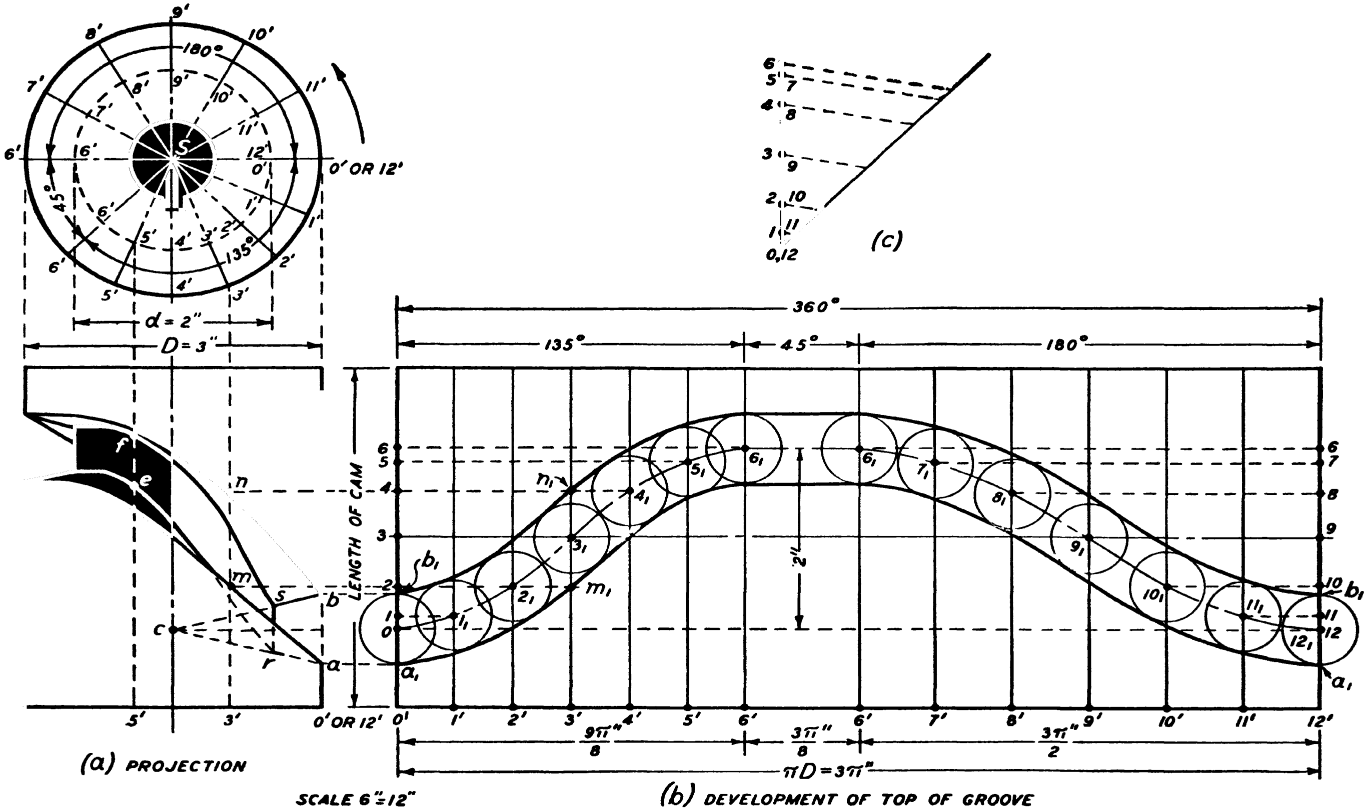
Depth of groove, ½ inch

Follower advances 2 inches in ⅜ turn of cam with U.A.R.M.

Follower rests during ⅛ turn of cam.

Follower returns 2 inches in ½ turn of cam with U.A.R.M.

Solution. Draw the plan and elevation of the cam cylinder as shown in Fig. 195(*a*), obtaining on the plan view two circles, one 3 inches in diameter to represent the outer surface of the cylinder in which the outer edges of the groove lie, and one 2 inches in diameter to represent that inner cylindrical surface at the bottom of the groove and in which of course the inner edges of the groove lie. Divide the plan, against the direction of rotation of the cam, into angular displacements of 135, 45, and 180 degrees respectively. Anticipating 6 follower positions for both the advance and return strokes, divide



(a) PROJECTION

SCALE 6" = 12"

(b) DEVELOPMENT OF TOP OF GROOVE

Fig. 195. Cylindrical Cam

both the 135 degree and the 180 degree angles into 6 equal divisions by radial lines of the plan, numbering the intersections of these radial lines and the circles from *1′* to *6′* and from *6′* to *12′* inclusive, as shown in the figure. The points so numbered are the horizontal projections or plan views of a series of elements of the cylindrical surfaces. Draw the vertical projections of these elements. Only two are shown in the figure: the projection from point *3′* of the *outer* circle, by which (as described hereafter) points n and m on the *outer* edges of the groove are located; and the projection from point *5′* of the *inner* circle, by which points f and e on the *inner* edges of the groove are located.

Draw the development of the outer surface of the cylinder in any convenient location, such as to the right of the elevation of the cylinder, so that points or distances can be transferred easily from the development to the projection drawing. This development is a rectangle whose base length is equal to the circumference of the base of the cylinder, and whose breadth is equal to the altitude of the cylinder. Its length is therefore equal to 3π inches; its breadth, 3¾ inches. Since the length of this development represents one turn of the cam, and hence 360 degrees of angular displacement, divide the base line into parts which are proportional to the angular displacements, 135, 45, and 180 degrees, of the given motion. Subdivide each one of these angular displacements on the base line in the same manner as it is subdivided in the plan view, numbering the points to agree with those of the plan. Then distance *0′-1′* on the base line is equal to arc *0′-1′* on the outer circle of the plan; distance *1′-2′* is equal to arc *1′-2′;* etc. Through these points on the base line draw vertical lines which are *elements* of the cylindrical surface.

Draw the horizontal center line of the development. Lay out along an element a distance equal to one-half the stroke of the follower above and below this center line. Divide this distance in agreement with the number of divisions of the angular displacements (see Fig. 195(*c*) and in accordance with the given motions during the advance and return strokes, to obtain positions of the follower. Number these positions from 1 to 6 on the advance and from 6 to 12 on the return stroke. Draw a horizontal line through each of these follower positions to intersect the corresponding element of the development at a point on the outer center line of the groove; for example, the horizontal line through point *1* intersects element $1'$ at point 1_1. Connect these points with a smooth curve to secure the development of the outer center line which from *0* to 6_1 is for the advance, from 6_1 to 6_1 is for the rest period, and from 6_1 to 12_1 is for the return. Now with a radius equal to one-half the width of groove, center the compass at points on the developed center line and describe circles as shown. Draw a smooth curve tangent to, and on either side of, these circles. These curves are the developed outlines of the outer edges of the groove, and the development is now complete in so far as its use for a template is concerned.

Now transfer the developed outlines of the outer edges of the groove to the front view of the projection drawing. In order to do this, it should be noted that the elements of the cylinder are in their true length on both the development and the front view (or elevation) of the projection drawing. Therefore distances on the development from points of intersection of the outline of the groove to the base line can be laid off on the corresponding elements in the front view and will thus locate the projection of the outer edges of the groove. This can be done either by projecting across (when the relative location is as shown in the figure) or by transferring these dimensions with the dividers. For example, intersections m_1 and n_1 on element $3'$ are projected across to the front view of element $3'$ to give points *m* and *n* respectively, and intersections a_1 and b_1 are projected to element $0'$ or $12'$, to give points *a* and *b*. Connect the points thus obtained by a smooth curve which is the projection of the outer edges of the groove. Only the visible part of the groove is shown in Fig. 195. One should note that, in transferring the groove from the development to the projection, the development is in a sense moved over until its element $0'$ coincides with element $0'$ of the projection, whereupon the development is wrapped around the cylinder in a direction opposite to that of the cam's rotation.

In order to secure the projections of the lower edges of the groove and thus complete the projection drawing, it is first necessary to obtain the width of the groove at the bottom. Locate a point midway between points *a* and *b* of the front view. Erect a perpendicular to element $0'$ at this point, intersecting the center line at *c*. Connect points *a* and *b* to point *c* by lines *ac* and *bc*. Draw a vertical line at a distance of ½ inch, the depth of the groove, from the outside element

O′, intersecting *ac* and *bc* at points *r* and *s* respectively. Then *rs* is the width of the bottom of the groove, and the trapezoid *absr* is the correct section of the groove to take a conical follower.

Since the lower or inner edges of the groove lie in a cylindrical surface whose diameter is *d*, a development of this surface is now drawn as shown in Fig. 196, and the layout of the center line of the groove is completed in the same manner as in the previous development. Now with a radius equal to one-half of *rs*, center the compass at points on this center line and describe circles to which the outlines of the inner edges of the groove are tangent. Transfer the points of intersection, where these developed edges cut the elements of the inner cylindrical surface, to the front view, as indicated at *r* and *s*, and *e* and *f* on element *5′*. Connect the points thus obtained to complete the projection drawing.

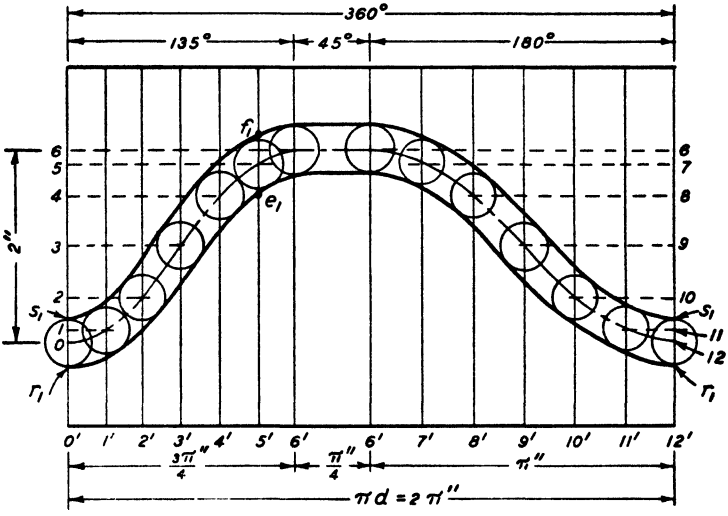


Fig 196. Development of Bottom of Groove

Cylindrical Cam Making Several Turns per Cycle. The prescribed motion of the follower of a cylindrical cam often requires the cam to make more than one turn during the advance of the follower and its return to the initial position; in other words, one cycle of the cam mechanism requires more than one turn of the cam.

The method of procedure in this design is exactly the same as that used with the preceding cylindrical cam, but it stresses one fact more forcibly. This fact is that the development of the groove goes across the development of the cylinder as many times as there are turns of the cam in one cycle, and each crossing of the development is in the same general direction.

This is clearly demonstrated by Fig. 197 which is shown drawn to scale, ⅓ size. It is the development of a cylindrical cam which gives to its follower the following motion:

Follower advances 4 inches in 1 turn of cam with S.H.M.

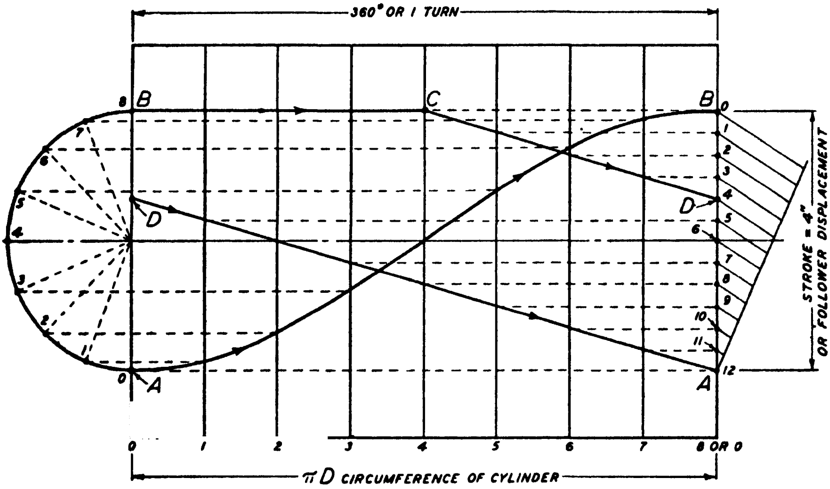


Fig. 197

Follower rests during ½ turn of cam.

Follower returns 4 inches (to initial position) in 1½ turns of cam with U.M.

It will be noted in the figure that the length of the development, πD, which represents one turn of the cam, is divided into 8 equal divisions. Each one of these divisions represents an angular displacement of 45 degrees. Hence, for the above motion, the groove must be developed through

8 of these divisions during the S.H.M.

4 of these divisions during the rest period, and

12 of these divisions during the U.M.

It is seen that the total number of divisions (time units) used is

8+4+12 or 24 divisions in all. This is of course equal to 24÷8 or 3 complete turns of the cam, which checks with the given motion. It is evident then that the distance traveled by the follower with simple harmonic motion must be divided to give 8 follower positions, while that with uniform motion must be divided to give 12 follower positions as seen in the figure. Points of intersections obtained on the development then afford a center line of groove which passes from

A to *B* in 8 units of time or 1 turn of cam

B to *C* in 4 units of time or ½ turn of cam, and

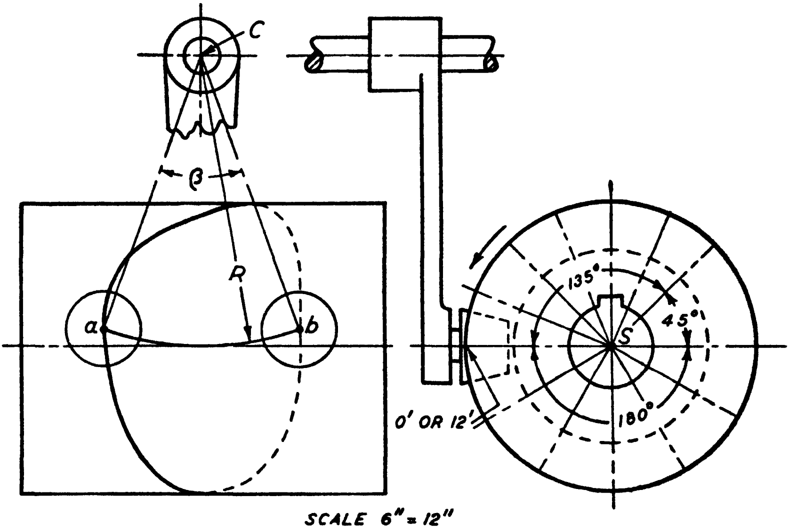


Fig 198. Cylindrical Cam with Oscillating Follower

C through *D* to the initial point *A* in 12 units of time or 1½ turns of cam.

In a cam with this motion it is evident that the groove has to cross itself. Due to this, the conical roller follower is replaced by an elongated slider whose head end will be well across the intersection and directed in the right path before its tail end arrives at the intersection.

Strap Cam. A cylindrical cam which is formed by strips of metal attached to the surface of the cylinder is called a *Strap Cam.* With some strap cams, the strips may be adjusted to different positions on the cylinder, thereby furnishing different motions to the follower.

Cylindrical Cam with Oscillating Follower. A cylindrical cam designed to work with an oscillating roller follower is shown in Fig. 198. The path of the center of the cam is along the arc *ab*, whose

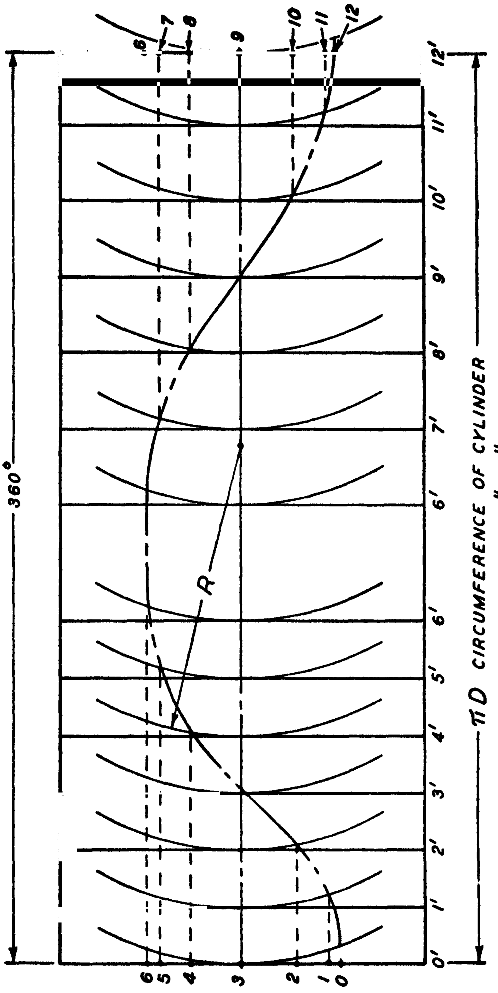


Fig. 199

radius, *R*, is the length of the arm which connects the follower to the pivot at *C*.

The center line of the groove is shown in Fig. 198, while the development from which the projection of the center line was obtained is shown in Fig. 199, the motion of the follower being the same as that used with the cam of Fig. 195. It will be noted that in the development of Fig. 199, arcs with centers on the center line of the development are described tangent to the elements of the cylinder. The radius with which these arcs are drawn is the radius, *R*, of the path *ab* of the follower. Intersections with these arcs (instead of with the elements as in Fig. 195) are points on, and hence determine the developed center line of, the groove. The center line thus obtained (or the edges of the groove obtained in the manner shown in Fig. 195) intersects the elements of the cylinder which may be transferred to the projection drawing in the usual manner. The method assumes the arc *ab* to be rather flat, and it takes for the follower displacement, *0-6* or *6-12* of Fig. 199, the projection of arc *ab* of Fig. 198 upon the center line or element of the cylinder.

In many cases the pivot *C* of Fig. 198 is moved up a trifle closer to the cam in order that the path *ab* may intersect the center line in such a manner that its mid-point is the same distance on one side as its ends are on the other. In such a case, the arcs in the development will be drawn to intersect the elements in the same manner.

QUESTIONS AND PROBLEMS*

1. Divide a line which is 3⁵⁄₁₆ inches in length into 8 divisions in accordance with uniform motion.

2. Divide a line which is 3 inches in length into 8 divisions in accordance with simple harmonic motion.

3. Divide a line which is 3 inches in length into 8 divisions in accordance with uniformly accelerated and retarded motion.

4. Obtain the profile or working surface of a disk cam to take a reciprocating knife-edge follower, the following information being available:

Rotation of cam, clockwise
Distance from center of cam shaft to edge of follower at initial position, 1¼ inches
Path of follower, along vertical center line of cam
Follower advances 1½ inches in ½ turn of cam with U.M.
Follower returns 1½ inches (to initial position) in ½ turn of cam with U.M.

5. Same as Problem 4 except that the motion on both strokes is S.H.M.

6. Same as Problem 4 except that the follower advances with S.H.M. and returns to its initial position with U.A.R.M.

7. Design a disk cam to work with a reciprocating roller follower, the following information being available:

Rotation of cam, counterclockwise
Path of follower, along vertical center line of cam
Diameter of follower, ¾ inch

---

*All graphical solutions of these problems to be made to scale full size.

Distance from center of cam to center of roller at initial position, 1½ inches

Follower advances 1⅜ inches in ⅜ turn of cam with U.A.R.M.

Follower returns 1⅜ inches (to initial position) in ⅜ turn of cam with U.A.R.M.

Follower rests during ¼ turn of cam.

8. Same as Problem 7 except cam turns clockwise and follower advances with S.H.M. and also returns to its initial position with S.H.M.

9. Design a disk cam to work with a reciprocating roller follower, the following information being available:

Rotation of cam, clockwise

Path of follower, along vertical center line of cam

Diameter of roller, ⅝ inch

Distance from center of cam to center of roller at initial position, 1¼ inches

Follower advances ⅝ inch in ¼ turn of cam with S.H.M.

Follower rests during ⅛ turn of cam.

Follower advances ⅝ inch in ¼ turn of cam with S.H.M.

Follower returns 1¼ inches (to initial position) in ⅜ turn of cam with U.A.R.M.

10. Draw the displacement diagram for a disk cam whose follower has the following motion. (Use a length of 6 inches.)

Follower advances 1½ inches in ⅜ turn of cam with U.M.

Follower rests during ¼ turn of cam.

Follower returns 1½ inches (to initial position) in ⅜ turn of cam with U.M.

11. Draw the displacement diagram for a disk cam whose follower has the following motion. Use a length of 6 inches.

Follower rests during ⅛ turn of cam.

Follower advances 1¼ inches in ¼ turn of cam with S.H.M.

Follower rests during ⅛ turn of cam.

Follower returns 1¼ inches (to initial position) in ½ turn of cam with U.A.R.M.

12. Draw the displacement diagram for a disk cam whose follower has the following motion. Use a length of 6 inches.

Follower advances ¾ inch in ¼ turn of cam with U.A.R.M.

Follower rests during 1/12 turn of cam.

Follower advances ¾ inch in ¼ turn of cam with U.A.R.M.

Follower returns 1½ inches (to initial position) in ⅓ turn of cam with U.A.R.M.

Follower rests during 1/12 turn of cam.

13. Design a disk cam to work with an offset knife-edge follower, whose path is along a right line which is ¾ inch to the left of the vertical center line of the cam. Further information is as follows:

Rotation of cam, clockwise

Initial position of follower, 1⅛ inches above horizontal center line of cam

Follower advances 1¼ inches in ¼ turn of cam with S.H.M.

Follower rests during ¼ turn of cam.

Follower returns 1¼ inches (to initial position) in ½ turn of cam with S.H.M.

14. Same as Problem 13 except follower is a ⅝-inch roller and motion on both advance and return strokes is U.A.R.M.

15. Design a disk cam to work with a reciprocating flat-face follower which moves along the vertical center line of the cam. The face of the follower is at right angles to the stem. Further information is as follows:

Rotation of cam, clockwise

Distance from center of cam to initial position of follower, 1½ inches

Follower advances 1 inch in ⅜ turn of cam with S.H.M.

Follower rests during ¼ turn of cam.

Follower returns 1 inch (to initial position) in ⅜ turn of cam with S.H.M.

16. What is a positive motion cam?

17. What is an involute cam?

18. Explain why a large pressure angle should be avoided in the design of a cam.

19. Does increasing the size of a cam increase or decrease the pressure angle?

20. Design a cam to work with an oscillating roller follower in accordance with the information given in the text for the cam of Fig. 185(*a*) and Fig. 185(*b*).

21. Same as Problem 20 except the motion is as follows:

Total angular displacement of follower, 25 degrees

Follower advances through an arc of 25 degrees in ⅜ turn of cam with U.A.R.M.

Follower rests during ¼ turn of cam.

Follower returns to initial position in ⅜ turn of cam with U.A.R.M.

22. From the information which is given in the text, design the cylindrical cam of Fig. 195.

23. Same as Problem 22 except that the motion is as follows:

Follower advances 2 inches in ⅓ turn of cam with S.H.M.

Follower rests during ⅙ turn of cam.

Follower returns 2 inches (to initial position) in ½ turn of cam with S.H.M.

24. Design the constant-breadth cam in accordance with the information given in the text for Fig. 188.

25. Same as Problem 24 except that the prescribed motion of the follower during the first half-turn is as follows:

Follower advances 1 inch in 135 degrees of rotation of cam with U.M.

Follower rests during 45 degrees of rotation of cam.

26. From the given data in the text reproduce the development of Fig. 197.

27. Obtain the development of a cylindrical cam showing the center line of the groove to meet the following conditions:

Length of cam, 5½ inches

Diameter of cylinder, 3¼ inches

Path of follower, parallel to axis of cam

Follower advances 3½ inches in 1¼ turns of cam with U.A.R.M.

Follower rests during ¼ turn of cam.

Follower returns 3½ inches (to initial position) in 1 turn of cam with U.M.

Follower rests during ½ turn of cam.

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